DOMINANCE RELATIONS ON THE SET OF SCHEDULES FOR UNCERTAIN JOB-SHOP

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Abstract. In the uncertain version of a job-shop problem $J/a_{ij} \leq p_{ij} \leq b_{ij}/F$ only lower and upper bounds of processing times (numerical input data) are assumed to be known before scheduling. We discuss dominance relations which prove to be useful to solve such a problem. A mixed graph is used for representing the structural input data (i.e. the precedence and capacity constraints), the scheduling process and the final solution. This mixed graph G defines the set of all feasible schedules (i.e. circuit-free digraphs) $\Lambda(G) = \{G_1, \ldots, G_\lambda\}$ via orienting all edges in G. Since the optimality of schedule G_s depends on the critical paths in G_s, in order to restrict the set of paths which may be critical we consider a dominance relation on the set of paths. Although this relation is based only on the structural input data, its use may considerably reduce the set of paths and hence simplify the mixed graph G and the digraphs $G_s \in \Lambda(G)$. To eliminate redundant schedules from the set $\Lambda(G)$, we generalize also the dominance relation due to the consideration of the numerical input data.

Key Words. Job shop, mixed graph, uncertainty.

1. INTRODUCTION

In contrast to well-studied deterministic settings, it is assumed that only the *structural* input data (i.e. the technological routes of all jobs) are known in advance while the processing time of each operation is uncertain and the probability distribution functions of the random processing times are unknown. Let n jobs $J = \{J_1, J_2, \ldots, J_n\}$ have to be processed on m machines $M = \{M_1, M_2, \ldots, M_m\}$ with specific routes (machine orders) of different jobs. Each machine can process at most one job at a time. The route of job $J_i \in J$ defines linearly ordered operations $O_{i1}, O_{i2}, \ldots, O_{in_i}$: At the stage $j \in \{1, 2, \ldots, n_i\}$ of job J_i, operation O_{ij} has to be processed on machine $M_{k_{ij}} \in M$. It is assumed that preemptions of an operation are forbidden: In any schedule, operation O_{ij} being started at time s_{ij} has to be processed up to its completion time $c_{ij} = s_{ij} + p_{ij}$, where p_{ij} denotes the processing time of operation O_{ij}.

We examine an uncertain version of a job-shop problem, where the structural input data (i.e. precedence and capacity constraints) are fixed before scheduling, but only a lower bound $a_{ij} \geq 0$, and an upper bound $b_{ij} \geq a_{ij}$ of the processing time of operation O_{ij}, $J_i \in J$; $j = 1, 2, \ldots, n_i$, are given: The actual processing time of operation O_{ij} may take any real value between these bounds. The objective is to find such a schedule which minimizes the given non-decreasing objective function $F(C_1, C_2, \ldots, C_n)$, where $C_i = c_{in_i}$ is the completion time of job $J_i \in J$. Using the three-field notation, this problem is denoted by $J/a_{ij} \leq p_{ij} \leq b_{ij}/F$. The problem $J/a_{ij} \leq p_{ij} \leq b_{ij}/F$ seems to be more realistic than the deterministic problem $J//F$: Even if there is no prior information on the possible perturbations of the processing times p_{ij}, one can consider 0 as the lower bound and a sufficiently large number as the upper bound of p_{ij}.

In [2, 3, 4] an approach for dealing with such 'strict uncertainty' based on a stability analysis of an optimal semiactive schedule (optimal digraph) was developed. In this paper we generalize this approach for any given regular criterion F.

2. PROBLEM SETTING

In an uncertain environment, it is not possible to determine a priori the starting times or completion times of operations. Let Q_k denote the set of operations which have to be processed on machine $M_k \in M$. So, the set of all operations Q may be represented as the union of the sets Q_k: $Q = \cup_{k=1}^{m} Q_k$, where $Q_k \cap Q_l = \emptyset$ with $k \neq l$. We define a set of m sequences $(O_{i_1k}, O_{i_2k}, \dots, O_{i_{|Q_k|}k})$ of operations Q_k, $k = 1, 2, \dots, m$, as a *schedule*. If none of these sequences contradicts others and the given precedence constraints, then for each fixed operation processing times these sequences uniquely define the *earliest starting* and *earliest completion times* of all operations Q, i.e. these sequences define a unique *semiactive* schedule.

To present the structural input data for problem $J/a_{ij} \leq p_{ij} \leq b_{ij}/F$, we use the mixed graph $G = (Q, A, E)$ with

- vertex set $Q = \{O_{ij} \mid J_i \in J;\ j = 1, 2, \dots, n_i\}$,
- arc set $A = \{(O_{ij}, O_{i,j+1}) \mid J_i \in J;\ j = 1, 2, \dots, n_i - 1\}$, and
- edge set $E = \{[O_{ij}, O_{uv}] \mid M_{k_{ij}} = M_{k_{uv}}; i \neq u\}$.

Let R_+^q denote the space of non-negative q-dimensional real vectors. Hereafter, q denotes the number of operations: $q = |Q| = \sum_{i=1}^{n} n_i = \sum_{k=1}^{m} |Q_k|$. Let T denote the polytope in the space R_+^q of all feasible vectors of processing times, i.e. $T = \{x = (x_{11}, x_{12}, \dots, x_{nn_n}) \mid a_{ij} \leq x_{ij} \leq b_{ij};\ i = 1, 2, \dots, n;\ j = 1, 2, \dots, n_i\}$. We define a *solution* to problem $J/a_{ij} \leq p_{ij} \leq b_{ij}/F$ as a set of schedules containing at least one optimal schedule for each feasible vector $p = (p_{11}, p_{12}, \dots, p_{nn_n})$ of processing times, i.e. for each vector $p \in T$. Thus, the whole set $\Lambda(G)$ is an obvious solution to problem $J/a_{ij} \leq p_{ij} \leq b_{ij}/F$ for each pair of vectors $a = (a_{11}, a_{12}, \dots, a_{nn_n}) \in R_+^q$ and $b = (b_{11}, b_{12}, \dots, b_{nn_n}) \in R_+^q$, where $a_{ij} \leq b_{ij}$; $i = 1, 2, \dots, n$; $j = 1, 2, \dots, n_i$. However, to construct the whole set $\Lambda(G)$ is only possible for a small problem size (since the cardinality λ of the set $\Lambda(G)$ could be equal to $2^{|E|}$). Therefore, it is practically important to look for a *minimal solution* of problem $J/a_{ij} \leq p_{ij} \leq b_{ij}/F$, i.e. for a minimal subset of set $\Lambda(G)$ containing at least one optimal schedule for each fixed vector $p \in T$ of processing times. We combine these definitions as follows.

DEFINITION 1: A set of schedules $\Lambda^*(G) \subseteq \Lambda(G)$ is a *solution* to problem $J/a_{ij} \leq p_{ij} \leq b_{ij}/F$ if for each fixed vector $p \in T$ of processing times the set $\Lambda^*(G)$ contains an optimal schedule. If any proper subset of the set $\Lambda^*(G)$ is no longer a solution to problem $J/a_{ij} \leq p_{ij} \leq b_{ij}/F$, it is called a *minimal solution* and we shall denote it by $\Lambda^T(G)$.

3. DOMINANCE RELATIONS

Let H_s^i denote the set of all paths in the digraph $G_s = (Q, A \cup A_s, \emptyset)$ ending in the *fixed* vertex O_{in_i} and starting from *different* vertices $O_{j1}, j = 1, 2, \dots, n$. The set of vertices (operations) which are contained in path μ will be denoted by $< \mu >$. Obviously, the value C_i for a schedule G_s is equal to the maximal length of a path from the set H_s^i, and hence, to solve problem $J//F$, we must find a schedule G_s such that $F_s^p = \min\{F_k^p \mid k = 1, 2, \dots, \lambda\}$, where $F_k^p = F(\max_{\nu \in H_k^1} l^p(\nu), \max_{\nu \in H_k^2} l^p(\nu), \dots, \max_{\nu \in H_k^n} l^p(\nu))$ is the value of the objective function of job completion times for schedule $G_k \in \Lambda(G)$ with fixed processing times $p \in R_+^q$ and $l^p(\mu)$ is the length of path μ: $l^p(\mu) = \sum_{O_{ij} \in <\mu>} p_{ij}$.

For a problem $J/a_{ij} \leq p_{ij} \leq b_{ij}/F$, we introduce the following transitive *dominance relation* which defines a partial ordering on the set of schedules $\Lambda(G)$.

DEFINITION 2: Schedule G_s *dominates* (strongly dominates) schedule G_k in domain $D \subseteq R_+^q$ if inequality $F_s^p \leq F_k^p$ (inequality $F_s^p < F_k^p$, respectively) holds for any vector $p \in D$ of processing times, and we shall denote the dominance relation by $G_s \preceq_D G_k$ (and the strong dominance relation by $G_s \prec_D G_k$).

If $a_{ij} = b_{ij}$ for each operation $O_{ij} \in Q$ (i.e. if T turns into a point which implies that problem $J/a_{ij} \leq p_{ij} \leq b_{ij}/F$ turns into a deterministic problem $J//F$), dominance relation $\preceq_T$ defines a total ordering on the set of schedules $\Lambda(G)$ and consequently the set $\Lambda^T(G)$ consists of a single schedule: $\Lambda^T(G) = \{G_s\}$, where G_s is any optimal schedule for problem $J//F$ with processing times p_{ij} being equal to $a_{ij} = b_{ij}$ for each operation $O_{ij} \in Q$. In other words, schedule G_s dominates all schedules $G_k \in \Lambda(G)$ in the point $a \in R_+^q$, i.e. $G_s \preceq_a G_k$. Moreover, if the strong dominance relation holds for each schedule $G_k \in \Lambda(G)$ in the point $a = b$, i.e. if $G_s \prec_a G_k$, then schedule G_s is the uniquely optimal one for the processing times p_{ij} equal to $a_{ij} = b_{ij}$.

For a problem $J/a_{ij} \leq p_{ij} \leq b_{ij}/F$, the operation processing times may vary between given lower and upper bounds and therefore it is a priori unknown which path from set H_k^i will have the maximal length in a practical realization of schedule G_k. Thus, we have to consider the whole set Ω_k^u of representatives of the family of sets $(H_k^i)_{J_i \in J}$. Each of these sets Ω_k^u includes exactly one path from each set H_k^i, $J_i \in J$. Since $H_k^i \cap H_k^j = \emptyset$ for any pair of different jobs J_i and J_j, we have the equality $|\Omega_k^u| = n$ and so there exist $\omega_k = \prod_{i=1}^n |H_k^i|$ different sets of representatives for digraph G_k, namely: $\Omega_k^1, \Omega_k^2, \ldots, \Omega_k^{\omega_k}$.

Next, we show how to restrict the number of sets of representatives which have to be considered while solving problem $J/a_{ij} \le p_{ij} \le b_{ij}/F$. Let $\Omega_k^{u^*}$, $u^* \in \{1, 2, \ldots, \omega_k\}$, denote the *critical set* in $G_k \in \Lambda(G)$ if the objective function value F_k^p is reached on this set of paths. For different vectors $p \in R_+^q$ of processing times, different sets Ω_k^u, $u \in \{1, 2, \ldots, \omega_k\}$, may be critical, however path $\nu \in H_k^i$, $J_i \in J$, may belong to a critical set only if $l^p(\nu) = \max_{\mu \in H_k^i} l^p(\mu)$. Therefore, while solving problem $J/a_{ij} \le p_{ij} \le b_{ij}/F$, it is sufficient to consider only paths from the set H_k^i which may have the maximal length for at least one vector $p \in T$ of processing times. Moreover, if there are two or more paths in H_k^i which have the maximal length for the same vector $p \in T$, it is sufficient to consider only one of them. Thus, it is sufficient to consider only *dominant* paths which are defined as follows.

DEFINITION 3: Path $\mu \in H_k^i, J_i \in J$, is dominant with respect to polytope T if for any path $\nu \in H_k^i$ system

$$\begin{cases} l^x(\mu) < l^x(\nu), \\ a_{ij} \le x_{ij} \le b_{ij}, \quad O_{ij} \in Q, \end{cases} \tag{1}$$

is inconsistent, where $x = (x_{11}, x_{12}, \ldots, x_{nn_n})$.

4. SOLUTION CHARACTERIZATION

Using the simple criterion for the consistency of system (1), which has been derived in [2], we can simplify digraph G_s while solving problem $J/a_{ij} \le p_{ij} \le b_{ij}/F$. Let $H_s^i(T)$ denote the set of all dominant paths in H_s^i with respect to polytope T. We denote by $G_s^T = (Q_s^T, A_s^T, \emptyset)$ the minimal subgraph of digraph G_s such that if $\mu \in \cup_{i=1}^n H_s^i(T)$, then digraph G_s^T contains path μ. To construct digraph G_s^T, one can use a modification of CPM [1].

A characterization of a solution Λ of problem $J/a_{ij} \le p_{ij} \le b_{ij}/F$ which is a proper subset of set $\Lambda(G)$, $\Lambda \subset \Lambda(G)$, may be obtained on the basis of the dominance relation $\preceq_D$ (see Definition 2).

THEOREM 1: The set $\Lambda \subset \Lambda(G)$ is a solution of problem $J/a_{ij} \le p_{ij} \le b_{ij}/F$ if and only if there exists a finite covering of polytope T by convex closed domains $D_j \subset R_+^q : T \subseteq \cup_{j=1}^d D_j$, $d \le |\Lambda|$, such that for any schedule $G_k \in \Lambda(G)$ and for any domain D_j, $j = 1, 2, \ldots, d$, there exists a schedule $G_s \in \Lambda$ such that dominance relation $G_s \preceq_{D_j} G_k$ holds.

Theorem 1 implies the following claim which characterizes a single-element solution of problem $J/a_{ij} \le p_{ij} \le b_{ij}/F$, which is necessarily a minimal solution.

COROLLARY 1: The equality $\Lambda^T(G) = \{G_s\}$ holds if and only if $G_s \preceq_T G_k$ for any schedule $G_k \in \Lambda(G)$.

LEMMA 1: Schedule $G_s \in B$ dominates schedule $G_k \in B$ in polytope T if (only if) the following inequality (2) holds (inequalities (3) hold, respectively):

$$F_s^b \le F_k^a \tag{2}$$

$$(F_s^a \le F_k^a,\ F_s^b \le F_k^b) \tag{3}$$

A minimal solution which includes more than one schedule may be characterized as follows.

THEOREM 2: Let set $\Lambda^*(G)$ be a solution of problem $J/a_{ij} \le p_{ij} \le b_{ij}/F$ with $|\Lambda^*(G)| > 1$. This solution is minimal if and only if for each schedule $G_s \in \Lambda^*(G)$ there exists a vector $p^{(s)} \in T$ such that inequality $F_s^{p^{(s)}} < F_k^{p^{(s)}}$ holds for each schedule $G_k \in \Lambda^*(G) \setminus \{G_s\}$.

In this paper, we focus on two types of dominance relations between feasible schedules (digraphs), which are useful for shop scheduling problems under strict uncertainty and regular criteria. Note that similar results for the C_{max} criterion have been derived in [2, 3, 4].

5. REFERENCES

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