# OPTIMAL COLORING OF THE VERTICES FROM THE UNIONS OF PATHS AND CLIQUES


**YURI SOTSKOV†, ALEXANDRE DOLGUI‡, FRANK WERNER***

† Institute of Engineering Cybernetics, Surganov St. 6, 220012 Minsk, Belarus, sotskov@newman.bas-net.by

‡ Laboratory of Industrial Systems Optimization, University of Technology of Troyes, 12, rue Marie Curie, BP 2060, 10010 Troyes cedex, France, dolgui@univ-troyes.fr

* Otto-von-Guericke-University, Faculty of Mathematics, PSF 4120, 39016 Magdeburg, Germany, frank.werner@mathematik.uni-magdeburg.de



**Abstract.** A unit-time scheduling problem with makespan criterion may be interpreted as a mixed graph coloring. This paper is devoted to an optimal coloring of a mixed graph G which defines a schedule minimizing makespan for the unit-time job-shop problem denoted by $J|p_{ij}=1|C_{max}$. We developed three branch-and-bound algorithms for an optimal coloring of a mixed graph G constructed for problem $J|p_{ij}=1|C_{max}$. These algorithms implemented in C++ were tested on randomly generated mixed graphs G of the order $n \leq 200$. The aim is to compare different bounds and branching schemes for coloring this type of mixed graphs.



**Key Words.** Scheduling algorithm, mixed graph, coloring.


## 1. PROBLEM SETTING AND NOTATIONS

Let G = (V, A, E) be a mixed graph with vertex set $V = \{v_1, v_2, ..., v_n\}$, arc set A, and edge set E. A mixed graph coloring $\psi$ may be defined as follows (see [1,3]). An integer-valued function $\psi : V \rightarrow \{1, 2, ..., t\}$ is a coloring of the mixed graph G = (V, A, E) if $\psi(v_i) < \psi(v_j)$ for each arc $(v_i, v_j) \in A$ and $\psi(v_p) \neq \psi(v_q)$ for each edge $[v_p, v_q] \in E$. The above coloring $\psi$ is optimal if it uses the minimum number $t = \gamma(G)$ of colors, and such a $\gamma(G)$ is called chromatic number of mixed graph G.

The minimization of the maximum completion time of n partially ordered operations $V = \{v_1, v_2, ... , v_n\}$ with unit (equal) processing times may be interpreted as an optimal coloring of the mixed graph (V, A, E), in which V is the given set of operations, the arc set A defines precedence constraints, and the edge set E defines capacity constraints. A coloring of such a mixed graph G defines a feasible assignment of operations V to the unit-time intervals: [0, 1], (1, 2], (2, 3], ..., (t-1, t].

To find an optimal coloring (i.e. one with minimal number $t = \gamma(G)$ of colors) of the given mixed graph (V, A, E) is an NP-hard problem even if $A = \varnothing$ [2]. Let $(V, \varnothing, E_A)$ denote a graph obtained from digraph $(V, A, \varnothing)$ by changing the set of arcs A by the set of edges $E_A = \{[v_i, v_j] : (v_i, v_j) \in A\}$. In [1], $O(n^2)$-algorithm has been developed for an optimal coloring of a mixed graph G for which graph $(V, \varnothing, E_A)$ is a tree. In [3,4], the chromatic polynomial and the chromatic number have been studied for a mixed graph coloring $\varphi$ for which inclusion $(v_i, v_j) \in A$ implies the non-strict inequality $\varphi(v_i) \leq \varphi(v_j)$.

Let the mixed graph G correspond to a unit-time, minimum-length, job-shop problem denoted by $J|p_{ij}=1|C_{max}$. Using graph terminology, one can note that the mixed graph G = (V, A, E) has the following two properties.

*(i)*: The partition $(V, \varnothing, E) = (V_1, \varnothing, E_1) \cup (V_2, \varnothing, E_2) \cup ... \cup (V_m, \varnothing, E_m)$ holds, where each graph $(V_i, \varnothing, E_i)$ is a clique for each i=1, 2, ..., m, and $V_i \cap V_k = \varnothing$ for $i \neq k$.

*(ii)*: The partition $(V, A, \varnothing) = (V^{(1)}, A^{(1)}, \varnothing) \cup (V^{(2)}, A^{(2)}, \varnothing) \cup \dots \cup (V^{(j)}, A^{(j)}, \varnothing)$ holds, where each digraph $(V^{(i)}, A^{(i)}, \varnothing)$ is a path for each $i = 1, 2, \dots, j$, and $V^{(i)} \cap V^{(k)} = \varnothing$ for $i \neq k$.

The above m and j denote the cardinality of the machine set $M = \{M_1, M_2, \dots, M_m\}$ and the cardinality of the job set $J = \{J_1, J_2, \dots, J_j\}$, respectively. Therefore, if $v_i \in V_k$, then operation $v_i$ has to be processed by machine $M_k \in M$, and vice versa. On the other hand, if inclusion $v_i \in V^{(k)}$ holds, then operation $v_i$ belongs to job $J_k \in J$, and vice versa. Thus, there exists a one-to-one correspondence between the mixed graph model G and the scheduling problem $J|p_{ij}=1|C_{max}$, namely: {vertex} ↔ {operation}, {path} ↔ {job}, {clique} ↔ {set of operations, which have to be processed by the same machine}. The complexity status of an optimal coloring of a mixed graph G with properties (i) and (ii) has been investigated in [5].

## 2. DESCRIPTION OF ALGORITHMS

We use the illustrative example of a mixed graph G presented in Fig. 1 to demonstrate the main ideas of the algorithms developed. This example corresponds to a problem $J5|n{=}4,p_{ij}{=}1|C_{max}$, in which job $J_1$ has to be processed by machines $M = \{M_1, M_2, M_3, M_4, M_5\}$ in the order $M_1, M_2, M_3, M_4, M_1, M_5$, job $J_2$ in the order $M_5, M_5, M_2, M_1, M_2, M_4$, job $J_3$ in the order $M_1, M_1, M_2, M_4, M_3, M_2$, and job $J_4$ in the order $M_2, M_1, M_3, M_2, M_1, M_5$. For simplicity, all edges and redundant arcs are omitted in Fig. 1.

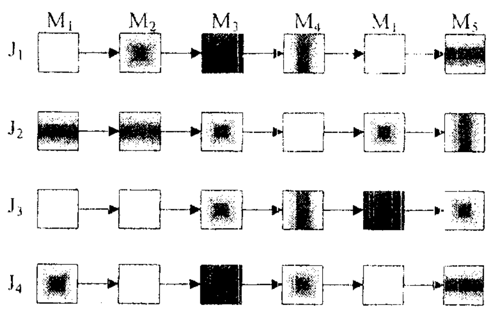


Fig. 1. An example of a mixed graph G (here edges are omitted).

The branch-and-bound algorithms will be determined via the description of a solution tree, the branching procedure, lower and upper bounds. For brevity, these elements of the algorithms are given using the above example of problem $J5|n{=}4,p_{ij}{=}1|C_{max}$. E.g., the solution tree for the first branching scheme is presented in Fig. 2 for the example $J5|n{=}4,p_{ij}{=}1|C_{max}$.

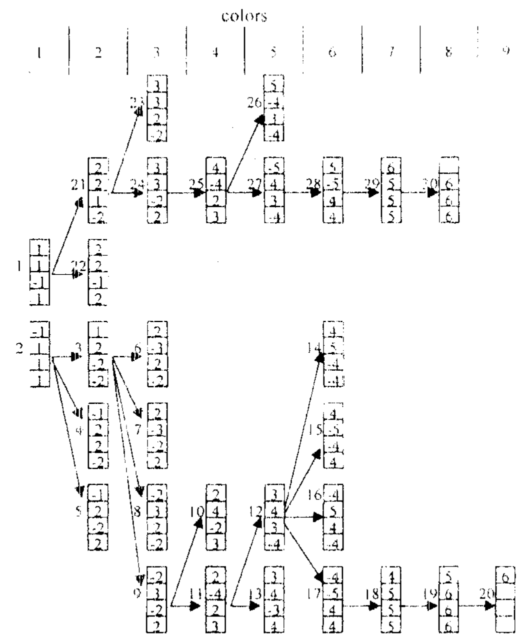


Fig. 2. Solution tree of algorithm GLOBAL1.

Each vertex $w^{(i)} \in W$ of a solution tree $T=(W,R)$ is a column with j elements. The first element $w_1^{(i)}$ of the vertex $w^{(i)} \in W$ corresponds to job $J_1 \in J$, the second element $w_2^{(i)}$ to job $J_2$, and so on, the last element $w_j^{(i)}$ to job $J_j$.

At the first iteration, each of all three algorithms tries to color a maximal possible number of operations from the set $V^{(1)} = \{v_{1_1}, v_{2_1}, \dots, v_{j_1}\}$ by the same minimal color 1, provided that this coloring does not cause any conflict due to possible edges existing in the mixed graph G.

For the example under consideration, the first operations of job $J_1$ and job $J_3$ cannot be colored by the same color 1 simultaneously, since these operations are connected by an edge (they have to be processed by the same machine $M_1$). Therefore, we construct two vertices $w^{(1)}$ and $w^{(2)}$ of the solution tree (it is a branching): in the first vertex $w^{(1)}$ the first operations of jobs $J_1$, $J_2$ and $J_4$ are colored by color 1, in the second vertex $w^{(2)}$ the first operations of jobs $J_2$, $J_3$ and $J_4$ are colored by color 1. The positions of the first operations of job $J_1$ in column $w^{(2)}$ and job $J_3$ in column $w^{(1)}$, which are not colored at the first iteration of the algorithm, are marked by −1 to indicate that the operation of stage 1 is not colored for these jobs. (See Fig. 2, where order numbers i of vertices $w^{(i)}$ are indicated at the left of the vertices.)

In general, each element $w_s^{(i)}$ of the column

$$w^{(i)} = \begin{bmatrix} w_1^{(i)} \\ w_2^{(i)} \\ \dots \\ w_j^{(i)} \end{bmatrix}$$

from the solution tree T is either equal to the order number of operation (i.e. to the order number of stage k) of job $J_s$ which is colored at this iteration of the algorithm, or element $w_s^{(i)}$ is equal to -k in the case when at this iteration no operation of the job $J_s$ is colored. The arc $(w^{(i)}, w^{(k)}) \in R$ of a solution tree T=(W,R) connects vertex $w^{(i)}$ with vertex $w^{(k)}$ if $w^{(k)}$ was generated from $w^{(i)}$ and for column $w^{(i)}$ color c was used, while for column $w^{(k)}$ color c+1 was used. In Fig. 2, the corresponding colors c used are shown at the top.

To overcome the conflict (when some vertices cannot be colored simultaneously by the same color due to the existence of a corresponding edge in the mixed graph), the algorithm uses branching of the set of possible colorings, i.e. the algorithm generates more than one vertex (instead of one vertex when there are no edges between operations which is ready to be colored and can be colored at this iteration). In general, the number of vertices generated in the solution tree T from the vertex $w_s^{(i)} \in W$ is equal to the product of the cardinalities of the sets of operations, where each set contains all operations, which may be colored at this iteration, but which cannot be colored simultaneously since they need the same machine to be processed. The terminal vertex of the solution tree T are either that which defines a coloring of all vertices of the mixed graph G or that for which the lower bound of the chromatic number $\gamma(G)$ is not less than the upper bound (UB) of the chromatic number $\gamma(G)$ being calculated earlier.

Note that in Fig. 2 vertex $w^{(20)}$ (and vertex $w^{(30)}$) have only one non-empty element (only three non-empty elements, respectively) while the other elements of these columns are empty. Such situations hold since at the previous iterations of the algorithm all operations of sets $V^{(2)}$, $V^{(3)}$ and $V^{(4)}$ of the jobs $J_2$, $J_3$ and $J_4$ (all operations of set $V^{(1)}$ of job $J_1$, respectively) have been colored.

Two lower bounds (a global $LB_1$ and a local $LB_2$) of the chromatic number $\gamma(G)$ have been tested in the experiments. The global lower bound is based on fixing the machine $M_k \in M$ and calculating the sum of the cardinality of the set $V_k$ and the minimum number $h^d_k$ (the minimum number $t^d_k$) of operations before the first operation (after the last operation), which needs machine $M_k$:

$$\gamma(G) \geq LB_1 = \max_{M_k \in M} \left\{ \min_{J_d \in J} h_k^d + |V_k| + \min_{J_d \in J} t_k^d \right\}. \quad (1)$$

The local lower bound $LB_2$ is very simple: it is equal to the maximum of the sums $k_i + l_i$ calculated for each job $J_i \in J$, where $k_i$ denotes the stage of job $J_i$ whose operation is not colored yet but is ready for coloring at the current iteration (i.e., the operation of job $J_i$ at stage $k_i - 1$ was colored at one of the previous iteration) and $l_i$ denotes the number of colors which were omitted for the operations of job $J_i$ at the previous iterations. Thus we have

$$\gamma(G) \geq LB_2 = \max_{J_i \in J} \{k_i + l_i\}. \quad (2)$$

At the initial iteration, the algorithms use the trivial upper bound $UB_0 = \sum_{i=1}^{J} |V_i|$. The number of colors used in the record coloring (i.e. in the best coloring) currently constructed is used as an upper bound UB of the chromatic number $\gamma(G)$. We coded three branch-and-bound algorithms depending on the lower bound used and on the selection of a vertex from the solution tree T for branching.

The depth-first search strategy was used in all three algorithms. The first and the second algorithms use the global lower bound $LB_1$ (see (1)). In the first algorithm (we call it GLOBAL1), the vertex $w^{(i)} \in W$ is selected from the set $W^*$ of all vertices generated at the current iteration if for this vertex $w^{(i)}$ the lower bound of $\gamma(G)$ has the minimum value among all the other vertices from the set $W^*$. If vertex $w^{(i)}$ defines a coloring of all the vertices V of the mixed graph G, then the algorithm selects vertex $w^{(r)}$ for the next branching, which has the minimum value of $LB_1$. If such a vertex $w^{(r)}$ is not uniquely determined, then one with the largest order number is selected from the whole set W for the next branching. The solution tree of algorithm GLOBAL1 for the example under consideration is presented in Fig. 2.

The second algorithm (we call it GLOBAL2) works as follows. If there are no terminal vertices in the solution tree among the vertices $W^*$ just generated, algorithm GLOBAL2 selects the vertex with minimal $LB_1$ among vertices $W^*$. If such a vertex $w^{(r)}$ is not uniquely determined, then one with the largest range (with the longest path from the root vertex to the vertex $w^{(r)}$) is selected for the next branching. If there exists a terminal vertex among the vertices $W^*$ just generated, algorithm GLOBAL2 selects the vertex with minimal lower bound less than $UB_0$. The solution tree of algorithm GLOBAL2 for the example under consideration is given in Fig. 3.

The third algorithm (we call it LOCAL) uses only $LB_2$ (see (2)) which is very fast for calculating, but usually worse than $LB_1$. Algorithm LOCAL uses the same rule as algorithm GLOBAL2 for selecting a vertex from the solution tree for branching. The solution tree of algorithm LOCAL for the example under consideration is given in Fig. 4. An optimal coloring for this example constructed by algorithm GLOBAL1 is given in Fig. 5.

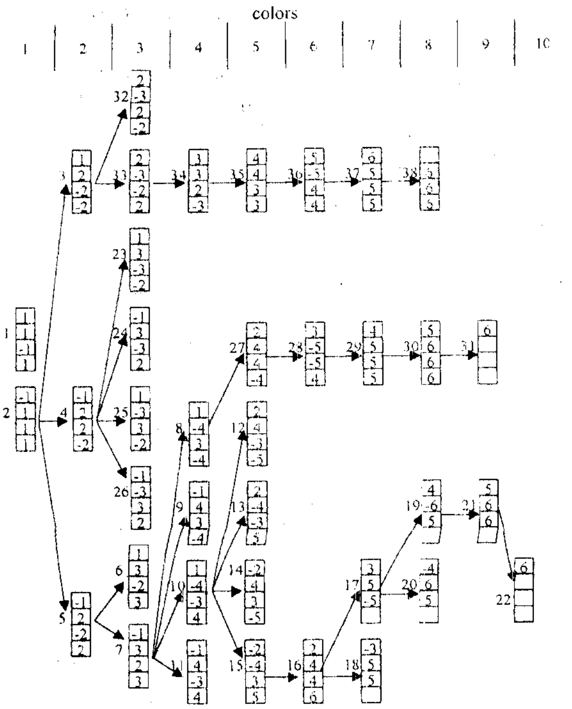


Fig. 3. Solution tree of algorithm GLOBAL2.

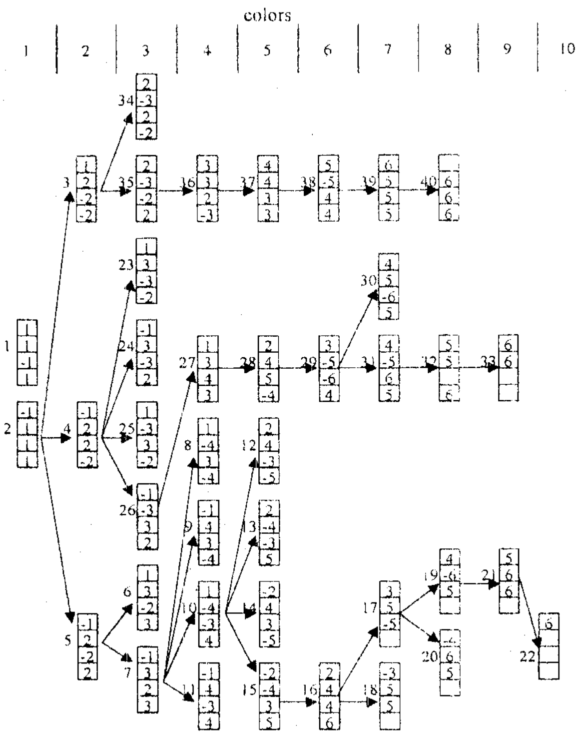


Fig. 4. Solution tree of algorithm LOCAL.

## 3. COMPUTATIONAL RESULTS

The above branch-and-bound algorithms have been implemented in C++ and tested on a PC Pentium II-350 with 133 MB RAM. In Tables 1 – 4, the computational results obtained for an optimal coloring of mixed graphs with properties (i) and (ii) are reported.

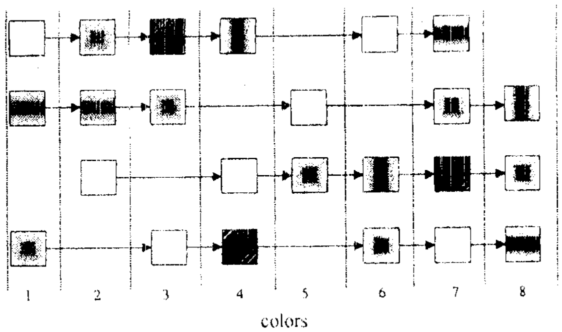


Fig. 5. An optimal coloring constructed by algorithm GLOBAL1.

The running times in seconds are presented in the last column of Tables 1, 2, 3 and 4 for pseudo-random mixed graphs G of orders n=120, n=150, n=180 and n=200, respectively. The first column in each table denotes the main parameters of the mixed graphs: number of cliques (the number of machines m), the number of paths (the number of jobs j), the length of each path (the number of stages $n_k$ per job $J_k$), the number |A| of arcs, and the number |E| of edges.

Each row in Tables 1 - 4 represents the results for 10 pseudo-random instances of series of mixed graphs with the same parameters m, j, $n_k$, |A| and |E|, the number $n_k$ of stages being the same for all jobs in an instance. The last column of the tables contains the average value of the running time for all 10 instances in the corresponding series. The second column contains the average first lower bound of the chromatic number constructed. The third column contains the average value of the chromatic number. The fourth column is equal to the percentage of problems for which the record coloring was proven to be optimal.

Table 1. Coloring of mixed graphs of order 120

| m, j, $n_k$, \|A\|, \|E\| | LB | $\gamma(G)$ | % | UB - LB | CPU |
|---|---|---|---|---|---|
| 10, 10, 12, 110, 772 | 19.5 | 19.7 | 100 | 0 | 5.764 |
| | 19.5 | 19.7 | 100 | 0 | 5.453 |
| | 19.5 | 19.9 | 80 | 1 | 74.018 |
| 11, 10, 12, 110, 649 | 17.8 | 18 | 100 | 0 | 6.490 |
| | 17.8 | 18 | 100 | 0 | 0.894 |
| | 17.8 | 18 | 100 | 0 | 6.975 |
| 12, 10, 12, 110, 601 | 17.3 | 17.6 | 100 | 0 | 1.363 |
| | 17.3 | 17.6 | 100 | 0 | 1.518 |
| | 17.3 | 17.6 | 100 | 0 | 22.747 |
| 13, 10, 12, 110, 545 | 16.3 | 17 | 100 | 0 | 3.098 |
| | 16.3 | 17 | 100 | 0 | 3.070 |
| | 16.3 | 17 | 100 | 0 | 8.996 |
| 14, 10, 12, 110, 516 | 16.4 | 16.8 | 100 | 0 | 0.313 |
| | 16.4 | 16.8 | 100 | 0 | 0.553 |
| | 16.4 | 16.8 | 100 | 0 | 0.859 |

Table 1 (continuation)

| | | | | | |
|---|---|---|---|---|---|
| 15, 10, 12, 110, 475 | 15.7 | 16 | 100 | 0 | 3.262 |
| | 15.7 | 16 | 100 | 0 | 3.174 |
| | 15.7 | 16 | 100 | 0 | 8.320 |
| 10, 12, 10, 108, 711 | 18.3 | 18.5 | 100 | 0 | 0.238 |
| | 18.3 | 18.5 | 100 | 0 | 0.233 |
| | 18.3 | 18.8 | 80 | 2 | 33.907 |
| 11, 12, 10, 108, 654 | 17 | 17.2 | 100 | 0 | 0.328 |
| | 17 | 17.2 | 100 | 0 | 0.327 |
| | 17 | 17.4 | 80 | 1.5 | 54.564 |
| 12, 12, 10, 108, 590 | 16.2 | 16.4 | 100 | 0 | 94.765 |
| | 16.2 | 16.4 | 100 | 0 | 93.988 |
| | 16.2 | 16.7 | 70 | 1 | 117.725 |
| 13, 12, 10, 108, 559 | 16.2 | 16.3 | 100 | 0 | 0.873 |
| | 16.2 | 16.3 | 100 | 0 | 0.894 |
| | 16.2 | 16.4 | 90 | 1 | 56.925 |
| 14, 12, 10, 108, 510 | 15.4 | 15.7 | 100 | 0 | 0.348 |
| | 15.4 | 15.7 | 100 | 0 | 0.346 |
| | 15.4 | 15.7 | 100 | 0 | 8.085 |
| 15, 12, 10, 108, 473 | 14.8 | 15 | 100 | 0 | 0.211 |
| | 14.8 | 15 | 100 | 0 | 0.205 |
| | 14.8 | 15 | 100 | 0 | 9.078 |

Table 2. Coloring of mixed graphs of order 150

| m, j, $n_k$, \|A\|, \|E\| | LB | γ(G) | % | UB - LB | CPU |
|---|---|---|---|---|---|
| 10, 10, 15, 140, 1103 | 21.9 | 22.2 | 100 | 0 | 573.013 |
| | 21.9 | 22.2 | 100 | 0 | 559.898 |
| | 21.9 | 23.1 | 50 | 2 | 131.793 |
| 11, 10, 15, 140, 1025 | 21.8 | 22.3 | 100 | 0 | 16.568 |
| | 21.8 | 22.3 | 100 | 0 | 31.201 |
| | 21.8 | 22.7 | 60 | 1.75 | 215.659 |
| 12, 10, 15, 140, 925 | 20.6 | 21.3 | 90 | 1 | 617.460 |
| | 20.6 | 21.3 | 90 | 1 | 619.555 |
| | 20.6 | 21.5 | 80 | 2 | 198.860 |
| 13, 10, 15, 140, 858 | 20 | 20.6 | 100 | 0 | 6.749 |
| | 20 | 20.6 | 100 | 0 | 11.068 |
| | 20 | 20.8 | 90 | 3 | 42.856 |
| 14, 10, 15, 140, 799 | 19.6 | 20.5 | 100 | 0 | 1.322 |
| | 19.6 | 20.5 | 100 | 0 | 1.439 |
| | 19.6 | 20.7 | 90 | 2 | 34.467 |
| 15, 10, 15, 140, 749 | 19.6 | 20.4 | 100 | 0 | 1.807 |
| | 19.6 | 20.4 | 100 | 0 | 6.740 |
| | 19.6 | 20.4 | 100 | 0 | 29.970 |
| 10, 15, 10, 135, 1109 | 20.9 | 20.9 | 100 | 0 | 33.529 |
| | 20.9 | 20.9 | 100 | 0 | 32.902 |
| | 20.9 | 21.5 | 60 | 1.5 | 42.010 |
| 11, 15, 10, 135, 1015 | 21.1 | 21.1 | 100 | 0 | 1.850 |
| | 21.1 | 21.1 | 100 | 0 | 1.796 |
| | 21.1 | 21.3 | 80 | 1 | 20.474 |
| 12, 15, 10, 135, 949 | 20.2 | 20.2 | 100 | 0 | 1.661 |
| | 20.2 | 20.2 | 100 | 0 | 1.547 |
| | 20.2 | 20.6 | 80 | 2 | 29.943 |
| 13, 15, 10, 135, 849 | 17.7 | 17.8 | 100 | 0 | 132.523 |
| | 17.7 | 17.8 | 100 | 0 | 130.629 |
| | 17.7 | 18.6 | 50 | 1.8 | 50.392 |

Table 2 (continuation)

| | | | | | |
|---|---|---|---|---|---|
| 14, 15, 10, 135, 791 | 17.5 | 17.5 | 100 | 0 | 30.182 |
| | 17.5 | 17.5 | 100 | 0 | 29.919 |
| | 17.5 | 18.1 | 50 | 1.2 | 70.000 |
| 15, 15, 10, 135, 747 | 17.1 | 17.1 | 100 | 0 | 8.820 |
| | 17.1 | 17.1 | 100 | 0 | 8.688 |
| | 17.1 | 17.4 | 80 | 1.5 | 26.035 |

Table 3. Coloring of mixed graphs of order 180

| m, j, $n_k$, \|A\|, \|E\| | LB | γ(G) | % | UB - LB | CPU |
|---|---|---|---|---|---|
| 10, 12, 15, 168, 1630 | 26.3 | 26.6 | 80 | 1 | 2040.980 |
| | 26.3 | 26.6 | 80 | 1 | 1728.010 |
| | 26.3 | 27.6 | 40 | 2.17 | 71.044 |
| 11, 12, 15, 168, 1472 | 24.2 | 24.3 | 100 | 0 | 63.250 |
| | 24.2 | 24.3 | 100 | 0 | 63.812 |
| | 24.2 | 25.7 | 10 | 1.67 | 165.704 |
| 12, 12, 15, 168, 1339 | 23.2 | 23.4 | 100 | 0 | 76.192 |
| | 23.2 | 23.4 | 100 | 0 | 79.425 |
| | 23.2 | 24.1 | 50 | 1.8 | 86.724 |
| 13, 12, 15, 168, 1242 | 21.9 | 22.5 | 100 | 0 | 350.082 |
| | 21.9 | 22.5 | 100 | 0 | 348.606 |
| | 21.9 | 23.5 | 40 | 2.5 | 239.662 |
| 14, 12, 15, 168, 1132 | 20.3 | 21.4 | 90 | 1 | 533.729 |
| | 20.3 | 21.4 | 90 | 1 | 544.346 |
| | 20.3 | 21.8 | 40 | 1.83 | 317.274 |
| 15, 12, 15, 168, 1065 | 20.6 | 21.1 | 100 | 0 | 680.937 |
| | 20.6 | 21.1 | 100 | 0 | 927.587 |
| | 20.6 | 21.6 | 60 | 1.75 | 122.328 |
| 10, 15, 12, 165, 1618 | 25.9 | 26 | 90 | 1 | 826.972 |
| | 25.9 | 26 | 90 | 1 | 812.931 |
| | 25.9 | 26.3 | 70 | 1.33 | 27.988 |
| 11, 15, 12, 165, 1482 | 23.9 | 24.1 | 80 | 1 | 855.919 |
| | 23.9 | 24.2 | 80 | 1.5 | 446.619 |
| | 23.9 | 25.4 | 20 | 1.88 | 65.840 |
| 12, 15, 12, 165, 1340 | 21.4 | 21.4 | 100 | 0 | 98.198 |
| | 21.4 | 21.4 | 100 | 0 | 102.636 |
| | 21.4 | 22.8 | 30 | 2 | 70.875 |
| 13, 15, 12, 165, 1243 | 22 | 22 | 100 | 0 | 5.764 |
| | 22 | 22 | 100 | 0 | 6.048 |
| | 22 | 22.4 | 70 | 1.33 | 28.948 |
| 14, 15, 12, 165, 1180 | 21.4 | 21.5 | 90 | 1 | 912.75 |
| | 21.4 | 21.5 | 90 | 1 | 923.122 |
| | 21.4 | 22.1 | 60 | 1.75 | 39.527 |
| 15, 15, 12, 165, 1081 | 19.6 | 19.6 | 100 | 0 | 81.585 |
| | 19.6 | 19.6 | 100 | 0 | 80.990 |
| | 19.6 | 20.4 | 40 | 1.33 | 73.548 |
| 13, 15, 12, 165, 1233 | 20.6 | 20.6 | 100 | 0 | 59.926 |
| | 20.6 | 20.6 | 100 | 0 | 60.274 |
| | 20.6 | 21.8 | 40 | 2 | 62.619 |
| 14, 15, 12, 165, 1149 | 20.4 | 20.5 | 90 | 1 | 894.761 |
| | 20.4 | 20.5 | 90 | 1 | 890.773 |
| | 20.4 | 21.2 | 50 | 1.6 | 63.110 |
| 15, 15, 12, 165, 1077 | 19.2 | 19.2 | 100 | 0 | 18.142 |
| | 19.2 | 19.2 | 100 | 0 | 19.365 |
| | 19.2 | 19.9 | 40 | 1.67 | 203.550 |

Table 4. Coloring of mixed graphs of order 200

| m, j, $n_k$, \|A\|, \|E\| | LB | γ(G) | % | UB - LB | CPU |
|---|---|---|---|---|---|
| 15, 10, | 24.2 | 26 | 100 | 0 | 629.115 |
| 20, 190, | 24.2 | 26 | 100 | 0 | 695.588 |
| 1320 | 24.2 | 26.6 | 20 | 2.7 | 422.098 |
| 16, 10, | 24.2 | 25.9 | 80 | 1 | 727.111 |
| 20, 190, | 24.2 | 25.9 | 80 | 1 | 923.124 |
| 1237 | 24.2 | 26.5 | 40 | 2.67 | 231.091 |
| 17, 10, | 24.1 | 25.6 | 90 | 2 | 424.019 |
| 20, 190, | 24.1 | 25.6 | 90 | 2 | 435.608 |
| 1188 | 24.1 | 25.7 | 90 | 3 | 105.709 |
| 18, 10, | 23.8 | 25.1 | 100 | 0 | 35.412 |
| 20, 190, | 23.8 | 25.1 | 100 | 0 | 36.667 |
| 1111 | 23.8 | 25.2 | 90 | 3 | 87.938 |
| 19, 10, | 22.8 | 24.2 | 100 | 0 | 18.205 |
| 20, 190, | 22.8 | 24.2 | 100 | 0 | 12.356 |
| 1041 | 22.8 | 24.2 | 100 | 0 | 19.221 |
| 20, 10, | 23.5 | 24.6 | 100 | 0 | 4.701 |
| 20, 190, | 23.5 | 24.6 | 100 | 0 | 8.812 |
| 983 | 23.5 | 24.6 | 100 | 0 | 20.639 |
| 21, 10, | 23 | 24 | 100 | 0 | 2.318 |
| 20, 190, | 23 | 24 | 100 | 0 | 1.198 |
| 936 | 23 | 24 | 100 | 0 | 8.914 |
| 22, 10, | 23 | 24.2 | 100 | 0 | 5.704 |
| 20, 190, | 23 | 24.2 | 100 | 0 | 6.224 |
| 905 | 23 | 24.4 | 90 | 3 | 44.038 |
| 23, 10, | 23.1 | 23.9 | 100 | 0 | 3.992 |
| 20, 190, | 23.1 | 23.9 | 100 | 0 | 4.020 |
| 860 | 23.1 | 23.9 | 100 | 0 | 7.789 |
| 24, 10, | 22.4 | 23.7 | 100 | 0 | 1.509 |
| 20, 190, | 22.4 | 23.7 | 100 | 0 | 1.243 |
| 829 | 22.4 | 23.7 | 100 | 0 | 2.654 |
| 25, 10, | 23.1 | 24.1 | 100 | 0 | 3.964 |
| 20, 190, | 23.1 | 24.1 | 100 | 0 | 2.885 |
| 810 | 23.1 | 24.1 | 100 | 0 | 3.252 |
| 15, 20, | 19.7 | 19.7 | 100 | 0 | 367.143 |
| 10, 180, | 19.7 | 19.7 | 100 | 0 | 357.141 |
| 1315 | 19.7 | 20.4 | 50 | 1.4 | 52.737 |
| 16, 20, | 19.7 | 19.8 | 90 | 1 | 1815.130 |
| 10, 180, | 19.7 | 19.8 | 90 | 1 | 1789.740 |
| 1234 | 19.7 | 20.4 | 40 | 1.17 | 53.496 |
| 17, 20, | 18.7 | 18.8 | 90 | 1 | 2383.930 |
| 10, 180, | 18.7 | 18.8 | 90 | 1 | 2233.300 |
| 1184 | 18.7 | 19.6 | 40 | 1.5 | 54.384 |
| 18, 20, | 18.9 | 19 | 90 | 1 | 4412.200 |
| 10, 180, | 18.9 | 19 | 90 | 1 | 4414.090 |
| 1093 | 18.9 | 19.6 | 60 | 1.75 | 38.255 |
| 19, 20, | 17.4 | 17.5 | 90 | 1 | 1867.810 |
| 10, 180, | 17.4 | 17.5 | 90 | 1 | 1683.440 |
| 1039 | 17.4 | 18.5 | 30 | 1.57 | 63.782 |
| 20, 20, | 17.4 | 17.8 | 70 | 1.33 | 4651.000 |
| 10, 180, | 17.4 | 17.8 | 70 | 1.33 | 4734.770 |
| 1003 | 17.4 | 18.5 | 30 | 1.57 | 64.046 |

Table 4 (continuation)

| | | | | | |
|---|---|---|---|---|---|
| 21, 20, | 17.3 | 17.4 | 100 | 0 | 118.445 |
| 10, 180, | 17.3 | 17.4 | 100 | 0 | 114.412 |
| 959 | 17.3 | 18.2 | 40 | 1.5 | 66.914 |
| 22, 20, | 16 | 16.2 | 90 | 1 | 1702.220 |
| 10, 180, | 16 | 16.2 | 90 | 1 | 1699.300 |
| 912 | 16 | 17.1 | 40 | 1.83 | 72.017 |
| 23, 20, | 16 | 16.1 | 100 | 0 | 13.023 |
| 10, 180, | 16 | 16.1 | 100 | 0 | 13.828 |
| 868 | 16 | 16.4 | 70 | 1.33 | 49.027 |
| 24, 20, | 15.7 | 15.8 | 100 | 0 | 48.133 |
| 10, 180, | 15.7 | 15.8 | 100 | 0 | 47.476 |
| 819 | 15.7 | 16.3 | 60 | 1.25 | 75.626 |
| 25, 20, | 17.1 | 17.1 | 100 | 0 | 1549.960 |
| 10, 180, | 17.1 | 17.1 | 100 | 0 | 1527.750 |
| 813 | 17.1 | 17.4 | 80 | 1.5 | 33.917 |

Each instance was solved by all three algorithms. The computational results for algorithm GLOBAL2 are represented in the first row of the three-row block, for algorithm GLOBAL1 in the second row, for algorithm LOCAL in the third row. If the fourth column is not equal to 100, it means that for at least one instance in this series, the upper limit L of the number of vertices W in a solution tree T = (W, R) was not sufficient to prove the optimality of the best coloring constructed. In the experiments reported, limit L was assumed to be equal to 20,000,000. The fifth column contains the average differences UB – LB for the 10 instances in the series. If for all 10 instances optimal colorings were constructed, then UB – LB = 0.

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