Calculation of the stability radius of an optimal line balance

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Abstract: For an assembly line, it is necessary to minimize the cycle time for processing a partially ordered set of operations $V = \{1, \ldots, n\}$ on a set of m linearly ordered working stations. The number m of stations and the initial processing times $t = (t_1, \ldots, t_n)$ of the operations V are given. However, for a subset $\widetilde{V} \subseteq V$ of the manual operations $j \in \widetilde{V}$, it is impossible to fix the processing times t_j for the whole life cycle of the assembly line. On the other hand, for each automated operation $i \in V \setminus \widetilde{V}$, the processing time t_i is fixed. We investigate the stability of an optimal line balance b_0 of the assembly line with respect to variations of the processing times $t_j, j \in \widetilde{V}$. It is shown how to calculate the stability radius $\rho_{b_0}(t)$ of an optimal line balance b_0, i.e., the maximal value of simultaneous independent variations of the processing times of the manual operations with keeping the optimality of the line balance b_0. We survey known results on the stability radius of an optimal line balance for a dual problem which is to minimize the number m of the working stations for the given cycle time.

Keywords: Optimal line balance; Stability analysis

1. INTRODUCTION

We consider a single-model paced assembly line (see Erel and Sarin (1998); Scholl (1999) for definitions and terminology) which continuously manufactures a homogeneous product in large quantities (a mass production).

An assembly line consists of a sequence of m linearly ordered (working) stations, which are linked by a conveyor belt. The set of operations V, which have to be repeatedly processed on the m stations, is fixed. Each operation $i \in V$ is indivisible. A station has to perform the same subset of operations of the set V within a cycle time c. All the m stations start simultaneously with the processing of a sequence of the operations, which are assigned to a station, and there is no buffer between sequential stations. The technological factors define a partial order on the set of operations V which is given by a digraph $G = (V, A)$, where V is the vertex set and A is the arc set. The set V includes two specific subsets of operations: the non-empty subset $\widetilde{V}$ includes all the manual operations, and the subset $V \setminus \widetilde{V}$ includes all the automated operations.

The simple assembly line balancing problem is to find an optimal line balance for a given set of m stations, i.e., to find a feasible assignment of the operations V to the m stations in such a way that the cycle time c reaches its minimal value. In Scholl (1999), the abbreviation SALBP-2 is used to denote such a problem.

Let $\widetilde{V} = \{1, \ldots, \tilde{n}\}$ and $V \setminus \widetilde{V} = \{\tilde{n} + 1, \ldots, n\}$, where $1 \leq \tilde{n} \leq n$. The vectors of the processing times of the operations are denoted as $\widetilde{t} = (t_1, \ldots, t_{\tilde{n}})$, $\overline{t} = (t_{\tilde{n}+1}, \ldots, t_n)$, and $t = (\widetilde{t}, \overline{t}) = (t_1, \ldots, t_n)$. Let the non-empty set $V_k^{b_r}$ of operations be assigned to station S_k, $k \in \{1, \ldots, m\}$. The assignment b_r

$$V = V_1^{b_r} \bigcup \ldots \bigcup V_m^{b_r}$$

of the operations V to m linearly ordered stations $S_1, \ldots, S_m$ (where

$$V_k^{b_r} \bigcap V_l^{b_u} = \emptyset$$

with $1 \leq k < l \leq m$) is called a line balance, if the following conditions 1 and 2 hold.

1. The assignment b_r does not violate the partial order given by the digraph $G = (V, A)$, i.e., inclusion $(i, j) \in A$ implies that operation $i \in V$ is assigned to station S_k and operation $j \in V$ is assigned to station S_l in such a way that inequalities $1 \leq k \leq l \leq m$ must hold.

2. The assignment b_r has to use all m stations:

$$V_k^{b_r} \neq \emptyset, \ k \in \{1, \ldots, m\}.$$

The line balance b_r is optimal if it provides the minimal cycle time c. The cycle time $c = c(b_r, t)$ for the line balance b_r with the vector t of the operation processing times is defined as

$$c(b_r, t) = \max_{k=1}^{m} \sum_{i \in V_k^{b_r}} t_i.$$

Let $B = \{b_0, \ldots, b_h\}$ be the set of all line balances. The optimality of the line balance $b_0 \in B$ with the vector t of the operation processing times is defined by the following condition:

3. $c(b_0, t) = \min\{c(b_r, t) : b_r \in B\}$

If $j \in \widetilde{V}$, the processing time t_j is a positive real number: $t_j > 0$. However, the value of the processing time t_j can vary during the life cycle of the assembly line and a modified processing time t'_j can be even equal to zero. A zero processing time $t'_j = 0$, $t'_j \neq t_j$, means that the manual operation

$$j \in V_k^{b_r} \bigcap \widetilde{V} := \widetilde{V}_k^{b_r}$$

is processed by an additional worker in parallel with processing other operations assigned to the station S_k. So, due to an additional worker, the processing of operation j does not increase the total processing time for the station S_k in the line balance b_r:

$$\sum_{i \in V_k^{b_r}} t'_i = \sum_{i \in V_k^{b_r} \setminus \{j\}} t'_i. \qquad (1)$$

It is clear that equality (1) is only possible if $t'_j = 0$.

If $j \in \widetilde{V}$, the processing time t_j is a real number which is fixed during the whole life cycle of the assembly line. One can assume that the strict inequality $t_i > 0$ holds for each automated operation $i \in V \setminus \widetilde{V}$ since an operation with a fixed zero processing time has no influence on a solution of the problem SALBP-2.

In contrast to a stochastic assembly line balancing problem (see the survey Erel and Sarin (1998)), we do not assume any probability distribution known in advance for the random processing time $t_j \in \widetilde{V}$.

Moreover, this paper does not deal with an algorithm for constructing an optimal line balance. It is assumed that an optimal line balance $b_0 \in B$ is somehow already constructed for the initial vector $t = (t_1, \ldots, t_n)$ of the operation processing times. Our aim is to investigate the stability of the optimality of a line balance b_0 with respect to simultaneous independent variations of the processing times of the manual operations $\widetilde{V}$.

In what follows, we show how to calculate the stability radius of an optimal line balance $b_0 \in B$. The stability radius may be interpreted as the maximum of simultaneous independent variations of the processing times of the manual operations $\widetilde{V}$ with definitely keeping the optimality of line balance b_0.

It should be noted that the processing times $t_i, i \in V$, are assumed to be real numbers (in contrast to the assumption used by Scholl (1999) that the processing times must be integers).

2. DEFINITION OF THE STABILITY RADIUS

The main question under consideration is as follows. How much can the components of a vector $\widetilde{t}$ simultaneously and independently be modified such that the line balance b_0, which is optimal for the initial vector t of the operation processing times, remains optimal for the modified vector t' of the operation processing times?

To be more precise, we study the stability radius of an optimal line balance, which is defined similarly to the stability radius of an optimal schedule (see Bräsel et al. (1996); Sotskov et al. (1997, 2010)).

If the stability radius of the line balance b_0 is strictly positive, then any simultaneous independent changes of the operation times $t_j, j \in \widetilde{V}$, within the ball with this radius and the center $\widetilde{t}$ definitely keep the optimality of the line balance b_0.

On the other hand, if the stability radius of the line balance b_0 is equal to zero, then even small changes of the processing times of the manual operations may deprive the optimality of the line balance b_0.

Let $R^{\tilde{n}}$ denote the space of real vectors $\widetilde{t} = (t_1, \ldots, t_{\tilde{n}})$ with the Chebyshev metric, i.e., the distance $d(\widetilde{t}, \widetilde{t}')$ between the vector $\widetilde{t}$ and the vector $\widetilde{t}' = (t'_1, \ldots, t'_{\tilde{n}})$ is defined as follows:

$$d(\widetilde{t}, \widetilde{t}') = \max\{|t_i - t'_i| : i \in \widetilde{V}\},$$

where $|t_i - t'_i|$ denotes the absolute value of the difference $t_i - t'_i$. Let $B(t)$ be the set of all line balances in the set B, which are optimal for the vector t of the operation processing times. $R_+^{\tilde{n}}$ denotes the space of non-negative real vectors.

Definition 1. The closed ball $O_\rho(\widetilde{t})$ in the space $R^{\tilde{n}}$ with the radius $\rho \in R_+^1$ and the center $\widetilde{t} \in R_+^{\tilde{n}}$ is called a stability ball of the line balance $b_0 \in B(t)$, if for each vector $t' = (\widetilde{t}', \overline{t})$ of the processing times with

$$\widetilde{t}' \in O_\rho(\widetilde{t}) \bigcap R_+^{\tilde{n}}$$

the line balance b_0 remains optimal. The maximal value $\rho_{b_0}(t)$ of the radius ρ of a stability ball $O_\rho(\widetilde{t})$ is called the stability radius of the line balance b_0.

In Definition 1, the initial vector $t = (\widetilde{t}, \overline{t})$ of the processing times is fixed, while the vector $\widetilde{t}'$ of the processing times of the manual operations may vary within the intersection of the closed ball $O_\rho(\widetilde{t})$ with the space $R_+^{\tilde{n}}$.

Let $W(b_r, t)$ be the set of subsets $\widetilde{V}_k^{b_r}, k \in \{1, \ldots, m\}$, for which equality (2) holds:

$$t(V_k^{b_r}) = c(b_r, t). \qquad (2)$$

Hereafter, we use the notation

$$t(V_k^{b_r}) = \sum_{i \in V_k^{b_r}} t_i,$$

where t indicates a vector $t = (\widetilde{t}, \overline{t}) = (t_1, \ldots, t_n) \in R_+^n$, for which the sum

$$\sum_{i \in V_k^{b_r}} t_i$$

has been calculated.

Sotskov et al. (2005) proved the following criterion for a zero value of the stability radius $\rho_{b_0}(t)$.

Theorem 1. Let $b_0 \in B(t)$. The equality $\rho_{b_0}(t) = 0$ holds if and only if there exists a line balance $b_s \in B(t)$ such that condition (3) does not hold:

$$W(b_0, t) \subseteq W(b_s, t). \qquad (3)$$

In Section 3, we present a sketch of the proof of Theorem 2 which allows us to calculate the stability radius $\rho_{b_0}(t)$ for the line balance $b_0 \in B(t)$ in the case when $\rho_{b_0}(t) \neq 0$.

3. CALCULATION OF THE STABILITY RADIUS

Assume that for a line balance $b_0 \in B(t)$, there does not exist a line balance $b_s \in B(t)$ such that inclusion (3) holds. Due to Theorem 1, the stability radius $\rho_{b_0}(t)$ is strictly positive. First, we consider the following case (j).

Case (j): $B(t) = \{b_0\}$

In order to calculate the exact value of a strictly positive stability radius $\rho_{b_0}(t) > 0$, we shall look for a line balance $b_r \in B \setminus \{b_0\}$ and for a vector $\widetilde{t^\epsilon} = (t_1^\epsilon, \ldots, t_{\tilde{n}}^\epsilon) \in R_+^{\tilde{n}}$ such that the following strict inequality holds:

$$c(b_r, t^\epsilon) < c(b_0, t^\epsilon), \qquad (4)$$

where $t^\epsilon = (\widetilde{t^\epsilon}, \bar{t})$ and the vector $\widetilde{t^\epsilon}$ may be as close as desired to the vector $\widetilde{t}$ provided that the strict inequality (4) holds.

Since the value $c(b_0, t)$ linearly depends on the components of the vector $\widetilde{t} = (t_1, \ldots, t_{\tilde{n}})$, before reaching the strict inequality in (4) via a continuous change of the components of the vector $\widetilde{t}$, one first reaches the following equality

$$c(b_r, t') = c(b_0, t')$$

for some vector $\widetilde{t'} = (t_1', \ldots, t_{\tilde{n}}')$, for which the optimal line balance b_0 becomes unstable, i.e., the equality

$$\rho_{b_0}(t') = 0 \qquad (5)$$

must hold.

Due to Theorem 1, from equality (5), it follows that there exists a line balance $b_r \in B(t')$ such that the condition $W(b_0, t') \subseteq W(b_r, t')$ does not hold. Hence, one can construct a vector $\widetilde{t^\epsilon}$, for which inequality (4) holds and $d(\widetilde{t^\epsilon}, \widetilde{t'}) = \epsilon$, where the positive number $\varepsilon \in R_+^1$ may be as small as desired.

Thus, the calculation of the stability radius $\rho_{b_0}(t)$ for the line balance $b_0 \in B(t)$ may be reduced to the construction of a vector $\widetilde{t'} \in R_+^{\tilde{n}}$, which is the closest one to the vector $\widetilde{t}$ provided that equality (5) holds.

In case (j), we have to compare the line balance b_0 with the non-optimal line balances $b_r \in B \setminus \{b_0\}$.

Let $t(V_k^{b_r}) > t(V_u^{b_0})$ and $\mid \widetilde{V}_u^{b_0} \oplus \widetilde{V}_k^{b_r} \mid \geq 1$.

Hereafter, the sign $\oplus$ denotes the direct summation of two sets as follows:

$$\widetilde{V}_u^{b_0} \oplus \widetilde{V}_k^{b_r} = \{\widetilde{V}_u^{b_0} \bigcup \widetilde{V}_k^{b_r}\} \setminus \{\widetilde{V}_u^{b_0} \bigcap \widetilde{V}_k^{b_r}\}.$$

It is easy to see that one will reach the equality

$$t'(V_k^{b_r}) = t'(V_u^{b_0}),$$

if the following value

$$\delta_{b_0,u}^{b_r,k} = \frac{t(V_k^{b_r}) - t(V_u^{b_0})}{\mid \widetilde{V}_u^{b_0} \oplus \widetilde{V}_k^{b_r} \mid} \qquad (6)$$

will be added (will be subtracted, respectively) to the processing time t_i for each operation $i \in \widetilde{V}_u^{b_o} \setminus \widetilde{V}_k^{b_r}$, i.e.,

$$t_i' = t_i + \delta_{b_0,u}^{b_r,k}$$

(from the processing time t_j for each operation $j \in \widetilde{V}_k^{b_r} \setminus \widetilde{V}_u^{b_0}$, i.e.,

$$t_j' = t_j - \delta_{b_0,u}^{b_r,k}).$$

It should be noted that in the latter modification, a component $t_j' = t_j - \delta_{b_0,u}^{b_r,k}$ of the vector $t' = (t_1', \ldots, t_n')$ may become negative.

In order to get a non-negative modified vector t', one has to test the operations of the set $\widetilde{V}_k^{b_r} \setminus \widetilde{V}_u^{b_0}$ in non-decreasing order of their processing times. Let $t_{(0)}^{uk}$ be equal to zero and let

$$t_{(0)}^{uk}, t_{(1)}^{uk}, \ldots, t_{(w_{uk})}^{uk}$$

denote a non-decreasing sequence of the processing times of the operations from the set $\widetilde{V}_k^{b_r} \setminus \widetilde{V}_u^{b_0}$, where $w_{uk} = |\widetilde{V}_k^{b_r} \setminus \widetilde{V}_u^{b_0}|$. A non-negative vector t' will be obtained if $\delta_{b_0,u}^{b_r,k}$ defined in (6) will be substituted by the following value:

$$\delta_{b_0,u}^{b_r,k} = \max_{\beta=0,\ldots,w_{uk}} \frac{t(V_k^{b_r}) - t(V_0^{b_u}) - \sum_{\alpha=0}^{\beta} t_{(\alpha)}^{uk}}{\max\{1, \mid \widetilde{V}_u^{b_0} \oplus \widetilde{V}_k^{b_r} \mid -\beta\}}. \qquad (7)$$

In order to obtain the minimal distance $d(t, t')$ between the desired vector t' and the initial vector t, one has to take the following maximum on k

$$\max_{t(V_k^{b_r}) > c(b_0,t)} \delta_{b_0,u}^{b_r,k},$$

then the following minimum on u

$$\delta(b_0, b_r, t) = \min_{|\widetilde{V}_u^{b_0} \oplus \widetilde{V}_k^{b_r}| \geq 1} \max_{t(V_k^{b_r}) > c(b_0,t)} \delta_{b_0,u}^{b_r,k}, \qquad (8)$$

and, finally, the following minimum

$$\delta(b_0, t) = \min_{b_r \in B \setminus \{b_0\}} \min_{|\widetilde{V}_u^{b_0} \oplus \widetilde{V}_k^{b_r}| \geq 1} \max_{t(V_k^{b_r}) > c(b_0,t)} \delta_{b_0,u}^{b_r,k}. \qquad (9)$$

Summarizing, we conclude that equality

$$\rho_{b_0}(t) = \delta(b_0, t)$$

must hold in the case (j).

Next, we consider the remaining possible case (jj).

Case (jj): $B(t) \setminus \{b_0\} \neq \emptyset$

Similarly as in the case (j), we have to calculate the value

$$\delta(b_0, t) = \min_{b_r \in B \setminus B(t)} \min_{|\widetilde{V}_u^{b_0} \oplus \widetilde{V}_k^{b_r}| \geq 1} \max_{t(V_k^{b_r}) > c(b_0,t)} \delta_{b_0,u}^{b_r,k}, \qquad (10)$$

where $\delta_{b_0,u}^{b_r,k}$ is defined in (7) and in contrast to (9), the first minimum is taken for all line balances $b_r \in B \setminus B(t)$.

In case (jj), along with calculating $\delta(b_0, t)$, we have to compare the line balance b_0 with all other optimal line balances $b_s \in B(t)$, $b_s \neq b_0$.

Let $\mid \widetilde{V}_u^{b_0} \oplus \widetilde{V}_k^{b_s} \mid \geq 1$ and

$$t(V_u^{b_0}) < c(b_s, t) = t(V_k^{b_s}),$$

where $b_s \in B(t) \setminus \{b_0\}$. It is easy to see that one will reach the equality

$$t'(V_k^{b_s}) = t'(V_u^{b_0}),$$

if the following value

$$\Delta_{b_0,u}^{b_s,k} = \frac{c(b_s,t) - t(V_u^{b_0})}{\mid \widetilde{V}_u^{b_0} \oplus \widetilde{V}_k^{b_s} \mid} \tag{11}$$

will be added (will be subtracted, respectively) to the processing time t_i for each operation $i \in \widetilde{V}_u^{b_0} \setminus \widetilde{V}_k^{b_s}$, i.e.,

$$t_i' = t_i + \Delta_{b_0,u}^{b_s,k}$$

(from the processing time t_j for each operation $j \in \widetilde{V}_k^{b_s} \setminus \widetilde{V}_u^{b_0}$, i.e.,

$$t_j' = t_j - \Delta_{b_0,u}^{b_s,k}).$$

Let $t_{(0)}^{uk}$ be equal to zero and let

$$t_{(0)}^{uk}, t_{(1)}^{uk}, \ldots, t_{(w_{uk})}^{uk}$$

denote a non-decreasing sequence of the processing times of the operations from the set $\widetilde{V}_k^{b_s} \setminus \widetilde{V}_u^{b_0}$, where $w_{uk} = |\widetilde{V}_k^{b_s} \setminus \widetilde{V}_u^{b_0}|$. A non-negative vector t' will be obtained if $\Delta_{b_0,u}^{b_s,k}$ defined in (11) will be substituted by the value $\Delta_{b_0,u}^{b_s,k}$ defined in (12):

$$\Delta_{b_0,u}^{b_s,k} = \max_{\beta=0,\ldots,w_{uk}} \frac{c(b_s,t) - t(V_u^{b_0}) - \sum_{\alpha=0}^{\beta} t_{(\alpha)}^{uk}}{\max\{1, \mid \widetilde{V}_u^{b_0} \oplus \widetilde{V}_k^{b_s} \mid -\beta\}}. \tag{12}$$

In order to obtain the minimal distance $d(t,t')$ of the desired vector t' to the initial vector t, one has to take the following maximum on k

$$\max_{t(V_u^{b_0}) < c(b_s,t) = t(V_k^{b_s})} \Delta_{b_0,u}^{b_s,k},$$

then the following minimum on u

$$\delta(b_0, b_s, t) = \min_{|\widetilde{V}_u^{b_0} \oplus \widetilde{V}_k^{b_s}| \geq 1} \max_{t(V_u^{b_0}) < c(b_s,t) = t(V_k^{b_s})} \Delta_{b_0,u}^{b_s,k}, \tag{13}$$

and, finally, the following minimum

$$\Delta(b_0, t) =$$

$$\min_{b_r \in B \setminus \{b_0\}} \min_{|\widetilde{V}_u^{b_0} \oplus \widetilde{V}_k^{b_s}| \geq 1} \max_{t(V_u^{b_0}) < c(b_s,t) = t(V_k^{b_s})} \Delta_{b_0,u}^{b_s,k}. \tag{14}$$

Summarizing, we conclude that equality

$$\rho_{b_0}(t) = \min\{\delta(b_0, t), \Delta(b_0, t)\}$$

must hold in the case (jj). So, for the general case, we can formulate the following theorem.

Theorem 2. If $b_0 \in B(t)$ and $\rho_{b_0}(t) \neq 0$, then

$$\rho_{b_0}(t) = \min\{\delta(b_0, t), \Delta(b_0, t)\},$$

where $\Delta(b_0, t)$ is defined in (14) and $\delta(b_0, t)$ is defined either in (9) or in (10).

Because of the page limit, we cannot present the complete proof of Theorem 2 (we are forced to remove some important details in the above argumentation).

The implementation of Theorem 2 is demonstrated on the calculation of the stability radius $\rho_{b_0}(t)$ for a small example given in Section 5.
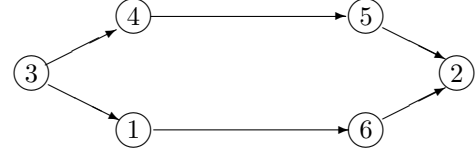

Fig. 1. The digraph $G = (V, A)$ defining a partial order on the set of operations V.

In Section 4, we show how one can restrict the number of line balances from the set $B \setminus \{b_0\}$, which have to be compared with the line balance b_0 for calculating the stability radius $\rho_{b_0}(t)$ using Theorem 2.

4. REDUNDANT LINE BALANCES FOR CALCULATING THE STABILITY RADIUS

Due to Theorem 2, the calculation of the stability radius of an optimal line balance is reduced to a time-consuming calculation on the set of line balances B. In the worst case, an optimal line balance b_0 must be compared with all the line balances from the set B. In order to restrict the number of line balances $b_r \in B \setminus \{b_0\}$ with which a comparison of the line balance b_0 has to be done for calculating the stability radius $\rho_{b_0}(t) > 0$, one can use the lower bound on the value $\delta(b_0, b_g, t)$ given in Lemma 3 which is presented without a proof.

We denote

$$\tilde{n}_{b_r} = \max\{|\widetilde{V}_k^{b_r}| : k \in \{1, \ldots, m\}\}.$$

Lemma 3. Let $b_0 \in B(t)$ and $b_r \in B \setminus \{b_0\}$. Then there is no need to compare the line balance b_0 with the line balance b_r if inequality

$$\delta(b_0, b_g, t) \geq \frac{c(b_r,t) - c(b_0,t)}{\min\{\tilde{n}, \tilde{n}_{b_r} + \tilde{n}_{b_0}\}} \tag{15}$$

holds for some line balance $b_g \in B \setminus \{b_0, b_r\}$.

The value $\delta(b_0, b_g, t)$ used in Lemma 3 is defined both in (9) and in (13).

In Lemma 3, it is assumed that the line balance b_0 is already compared with the line balance b_g. As a result, the following upper bound on the stability radius $\rho_{b_0}(t)$ is obtained:

$$\rho_{b_0}(t) \leq \delta(b_0, b_g, t).$$

Due to Lemma 3, the line balance $b_r \in B \setminus \{b_0\}$ may make precise the achieved volume of the stability ball $O_\rho(\tilde{t})$ only if inequality (15) does not hold. Thus, the line balances $b_r \in B \setminus \{b_0\}$ must be compared with the line balance b_0 in the order of increasing the right-hand side of inequality (15).

5. EXAMPLE

Let $m = 3$, $n = 6$, $\tilde{n} = 3$ and $t = (\tilde{t}, \bar{t}) = (4, 1, 6, 3, 7, 4)$. The set $\widetilde{V} = \{1, 2, 3\}$ includes all the manual operations. The set $V \setminus \widetilde{V} = \{4, 5, 6\}$ includes all the automated operations. The digraph $G = (V, A)$ defining a partial order on the set of operations V is presented in Fig. 1.

If all the processing times are integers, then the following lower bound on the minimal cycle time may be used:

$$\min\{c(b_r,t) : b_r \in B\} \geq \left\lceil \sum_{i=1}^{n} \frac{t_i}{m} \right\rceil, \qquad (16)$$

where $\lceil a \rceil$ denotes the smallest integer greater than or equal to a.

One can convince that the line balance b_0

$$V = \{3,4\}\bigcup\{1,6\}\bigcup\{2,5\}$$

is optimal. Indeed, for the line balance b_0, we obtain

$$\left\lceil \sum_{i=1}^{n} \frac{t_i}{m} \right\rceil = \left\lceil \sum_{i=1}^{6} \frac{t_i}{3} \right\rceil = \left\lceil \frac{26}{3} \right\rceil = 9 = c(b_0,t),$$

which implies that b_0 is an optimal line balance since $c(b_0,t)$ is equal to the right-hand side of inequality (16).

Similarly, one can convince that the line balance b_1

$$V = \{3,4\}\bigcup\{5\}\bigcup\{1,2,6\}$$

is also optimal. The other line balances are non-optimal for the vector $t = (\widetilde{t},\bar{t}) = (4,1,6,3,7,4)$ of the operation processing times.

For the line balance b_1, there exists a line balance $b_0 \in B(t)$ such that condition (3) does not hold, namely:

$$\{\{3\},\{1,2\}\} = W(b_1,t) \not\subseteq W(b_0,t) = \{\{3\}\}.$$

So, due to Theorem 1, we obtain the equality

$$\rho_{b_1}(t) = 0$$

which means that the optimality of the line balance $b_1 \in B(t)$ is unstable: even small changes of the processing times of the manual operations $1, 2$ and 3 may deprive the optimality of the line balance b_1.

On the other hand, for the line balance b_0, we obtain

$$\{\{3\}\} = W(b_0,t) \subseteq W(b_1,t) = \{\{3\},\{1,2\}\},$$

where $\{b_1\} = B(t) \setminus \{b_0\}$. So, due to Theorem 1, the strict inequality

$$\rho_{b_0}(t) > 0$$

must hold, and the optimality of the line balance $b_0 \in B(t)$ is stable: there exists a closed ball $O_\rho(\widetilde{t})$ in the space R^3 with the positive radius $\rho \in R_+^1$ and the center $\widetilde{t} \in R_+^3$ such that inclusion $b_0 \in B(t')$ holds for any modified vector $t' = (\widetilde{t'},\bar{t})$ of the operation processing times with

$$\widetilde{t'} \in O_\rho(\widetilde{t})\bigcap R_+^3.$$

Due to Theorem 2, we can calculate the maximum value $\rho_{b_0}(t)$ of such a radius ρ.

First, we compare the line balance b_0 with the line balance b_1, which is also optimal for the initial vector $t = (4,1,6,3,7,4)$: $b_1 \in B(t)$. Using formula (13), we calculate

$$\delta(b_0,b_1,t) = 1.$$

Using formula (14), we obtain $\Delta(b_0,t) = 1$.

Using bound (15) with the left-hand side $\delta(b_0,b_1,t)$ equal to 1, we can convince that, to calculate the value $\delta(b_0,t)$ given in (9), it is sufficient to compare the line balance b_0 with five line balances as follows:

with the line balance b_2
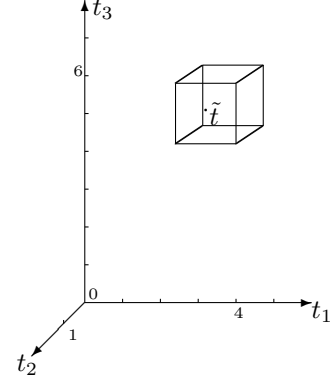
$$V = \{3,4\}\bigcup\{1,5\}\bigcup\{2,6\},$$

Fig. 2. The stability ball $O_\rho(\tilde{t})$ with the radius $\rho = \rho_{b_0}(t) = \frac{2}{3}$ and the center $\tilde{t} = \{4,1,6\}$ for the line balance $b_0 \in B(t) = \{b_0,b_1\}$.

with the line balance b_3

$$V = \{1,3\}\bigcup\{4,5\}\bigcup\{2,6\},$$

with the line balance b_4

$$V = \{1,3\}\bigcup\{4,6\}\bigcup\{2,5\},$$

with the line balance b_5

$$V = \{1,3\}\bigcup\{6\}\bigcup\{2,4,5\},$$

and with the line balance b_6

$$V = \{3\}\bigcup\{4,5\}\bigcup\{1,2,6\}.$$

Using formula (8), we calculate

$$\delta(b_0,b_2,t) = 1,$$
$$\delta(b_0,b_3,t) = 1,$$
$$\delta(b_0,b_4,t) = \frac{2}{3},$$
$$\delta(b_0,b_5,t) = \frac{2}{3},$$
$$\delta(b_0,b_6,t) = 1.$$

Using formula (9), we obtain

$$\delta(b_0,t) = \min\left\{1,1,\frac{2}{3},\frac{2}{3},1\right\} = \frac{2}{3}.$$

Using Theorem 2, we obtain

$$\rho_{b_0}(t) = \min\{\delta(b_0,t),\Delta(b_0,t)\} = \min\left\{\frac{2}{3},1\right\} = \frac{2}{3}.$$

The stability ball $O_\rho(\tilde{t})$ with the radius

$$\rho = \rho_{b_0}(t) = \frac{2}{3}$$

and the center $\tilde{t} = \{4,1,6\}$ for the line balance $b_0 \in B(t)$ is presented in Fig. 2.

6. A SURVEY OF STABILITY RESULTS FOR A DUAL PROBLEM

At the stage of the design of the assembly line, the following problem, which is dual to the problem SALBP-2, has to be solved: minimize the number of stations m for processing a set of partially ordered operations $V = \{1,\ldots,n\}$ within the given cycle time c.

The assignment b_r

$$V = V_1^{b_r} \bigcup \ldots \bigcup V_m^{b_r}$$

of the operations V to m linearly ordered stations $S_1, \ldots, S_m$ (where $V_k^{b_r} \bigcap V_l^{b_u} = \emptyset$ with $1 \leq k < l \leq m$) is a feasible line balance, if condition 1 (see Section 1) and the following condition 2^* hold.

2^*. The given cycle time c is not violated for each station S_k, $k \in \{1, \ldots, m\}$, i.e., the station time $t(V_k^{b_r})$ has to be no greater than the cycle time c.

The feasible line balance $b_0 \in B$ is optimal if it provides the minimal number $m = m_{b_0}$ of stations used, i.e., the following condition 3^* holds.

3^*. $m = m_{b_0} = \min\{m(b_r) : b_r \in B\}$

Scholl (1999) uses the abbreviation SALBP-1 to denote such a dual problem.

In practice, at the design stage of an assembly line, the cycle time c may be calculated on the basis of the customer demands for the finished products, which will be assembled on the designed assembly line.

Similarly to the problem SALBP-2, considered in Sections 1–5, problem SALBP-1 is binary NP-hard even for the case of two stations ($m = 2$) used in the optimal line balance.

A stability analysis for the problem SALBP-1 was developed by Sotskov and Dolgui (2001); Sotskov et al. (2006) with slightly different definitions of the stability radii. In particular, Sotskov et al. (2006) use the following definition.

Definition 2. For the problem SALBP-1, the open ball $O_\rho^1(\tilde{t})$ in the space $R^{\tilde{n}}$ with the radius $\rho \in R_+^1$ and the center $\tilde{t} \in R_+^{\tilde{n}}$ is called a stability ball of the line balance $b_0 \in B(t)$, if for each vector $t' = (\tilde{t}', \bar{t})$ of the processing times with

$$\tilde{t}' \in O_\rho^1(\tilde{t}) \bigcap R_+^{\tilde{n}}$$

the line balance b_0 remains feasible and optimal. The maximal value $\rho_{b_0}^1(t)$ of the radius ρ of a stability ball $O_\rho^1(\tilde{t})$ is called the stability radius of the line balance b_0 for the problem SALBP-1.

The criterion (i.e., the necessary and sufficient condition) for the equality $\rho_{b_0}^1(t) = 0$ and a general formula for calculating the stability radius $\rho_{b_0}^1(t)$ have been derived by Sotskov et al. (2006).

In the definition of the stability radius of an optimal line balance for the problem SALBP-1, used by Sotskov and Dolgui (2001), it is assumed that the open ball $O_\rho^1(\tilde{t})$ is totally located within the space $R_+^{\tilde{n}}$. Therefore, in Sotskov and Dolgui (2001), generally smaller stability radii may be obtained. In particular, due to the definition of the stability radius of an optimal line balance used by Sotskov and Dolgui (2001), the stability radius of an optimal line balance cannot be grater than

$$\min\{t_i : i \in \widetilde{V}\}.$$

So, Definition 2 is more appropriate for a practical use than that introduced by Sotskov and Dolgui (2001).

7. CONCLUSION

Along with using the lower bound on the value $\delta(b_0, b_g, t)$ given in Lemma 3, we can restrict further the number of line balances $b_r \in B \setminus \{b_0\}$ with which a comparison of the line balance b_0 has to be done for calculating the stability radius $\rho_{b_0}(t) \neq 0$.

In particular, one does not need to distinguish the line balances, which have different orders of the subsets in

$$V = V_1^{b_r} \bigcup \ldots \bigcup V_m^{b_r},$$

but their sets of subsets

$$\{V_1^{b_r}, \ldots, V_m^{b_r}\}$$

are the same.

Note also, that in practice not all line balances are suitable for a realization in the assembly line since not only precedence constraints defined by the set of arcs A have to be taken into account. Due to this, the cardinality of the set of line balances tested in Theorem 2 and in Lemma 3 may be considerably smaller than the cardinality of the set B.

It is easy to see that problem SALBP-2 with n unordered operations V (i.e., with digraph $G = (V, \emptyset)$) and m working stations, may be interpreted as the problem of optimal scheduling n unordered jobs on m parallel identical machines with the makespan criterion. Therefore, the results presented in Sections 1–5 on the stability analysis of an optimal line balance for the problem SALBP-2 with n unordered operations and m working stations may be interpreted as new results on the stability analysis of a schedule minimizing the makespan for processing n unordered jobs on m parallel identical machines.

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