The Optimality Box and Region for Single-Machine Scheduling of a Set of Jobs with Uncertain Durations

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Problem setting

$r_i = 0$ – release time.

$J = \{J_1, J_2, ..., J_n\}$ – set of jobs to be processed.

$p_i \in [p_i^L, p_i^U]$ – processing time of job J_i.

Preemptions of operations are not allowed.

The exact value of the duration of a job will be known after the completion of this job. The objective criterion is:

$$\sum_{i=1}^{n} C_i \to \min \quad (1)$$

(C_i – the completion time of job J_i)

$$1 \mid p_i^L \le p_i \le p_i^U \mid \sum C_i \quad (2)$$

If $p_i^L = p_i^U$ for all jobs J_i, then an instance $1 \| \sum C_i$ can be solved in $O(n \log n)$ time*.

*Smith W.E. Various Optimizers for Single-Stage Production. – Naval Research and Logistics Quarterly. – 1956. – 3, №1. – P. 59 – 66.

A duration vector $p \in T$ – ***scenario***

$T = \{p \mid p \in R^n_+,\ p_i^L \le p_i \le p_i^U,\ i \in \{1, \ldots, n\}\} = \times_{i=1}^{n} [p_i^L, p_i^U]$ - the set of all scenarios.

$1 | p | \sum C_i$ – deterministic problem with a fixed scenario $p \in T$

$\Pi = \{\pi_1, \ldots, \pi_{n!}\}$

Theorem 1. A permutation $\pi_k = (J_{k_1}, J_{k_2}, \ldots, J_{k_n}) \in \Pi$ of the jobs J is optimal for the instance $1 | p | \sum C_i$ if and only if

$$p_{k_1} \le \ldots \le p_{k_n}$$

RELATED RESULTS: **The Optimality Box**

$M=(k_{i_1},k_{i_2},...,k_{i_{|M|}}),\ k_{i_1}<k_{i_2}<...<k_{i_{|M|}}$ - ordered subset of the set $\{1,2,...,n\}$

Definition 1. The maximal (with respect to inclusion) rectangular box (3)

$$OB(\pi_k,T)=[l^{opt}_{k_{i_1}},u^{opt}_{k_{i_1}}]\times[l^{opt}_{k_{i_2}},u^{opt}_{k_{i_2}}]\times...\times[l^{opt}_{k_{i_{|M|}}},u^{opt}_{k_{i_{|M|}}}]\subseteq T$$

is called the **optimality box** for the permutation $\pi_k=(J_{k_1},J_{k_2},...,J_{k_n})\in\Pi$ with respect to T, if the permutation π_k being optimal for the instance $1|p|\sum C_i$ with the scenario $p=(p_1,p_2,...,p_n)\in T$ remains optimal for the instance $1|p'|\sum C_i$ with any scenario (4)

$$p'\in[p_1,p_1]\times[p_2,p_2]\times...\times[p_{i_g-1},p_{i_g-1}]\times[l^{opt}_{i_g},u^{opt}_{i_g}]\times[p_{i_g+1},p_{i_g+1}]\times...\times$$

$\times[p_n,p_n]$.If there does not exist a scenario $p\in T$ such that the permutation π_k is optimal for the instance $1|p|\sum C_i$, then $OB(\pi_k,T)=\varnothing$.

$[l^{opt}_{k_r},u^{opt}_{k_r}],\ l^{opt}_{k_r}\le u^{opt}_{k_r}$, - **segment of optimality** for the job $J_{k_r}\in J$

The Optimality Region

Definition 2. The maximal closed region $OR(\pi_k, T) \subseteq T$ in the set R_+^n is called the **optimality region** for the permutation $\pi_k = (J_{k_1}, J_{k_2}, \ldots, J_{k_n}) \in \Pi$ with respect to T, if the permutation π_k is optimal for the instance $1|p|\sum C_i$ with any scenario $p = (p_1, p_2, \ldots, p_n) \in OR(\pi_k, T)$.

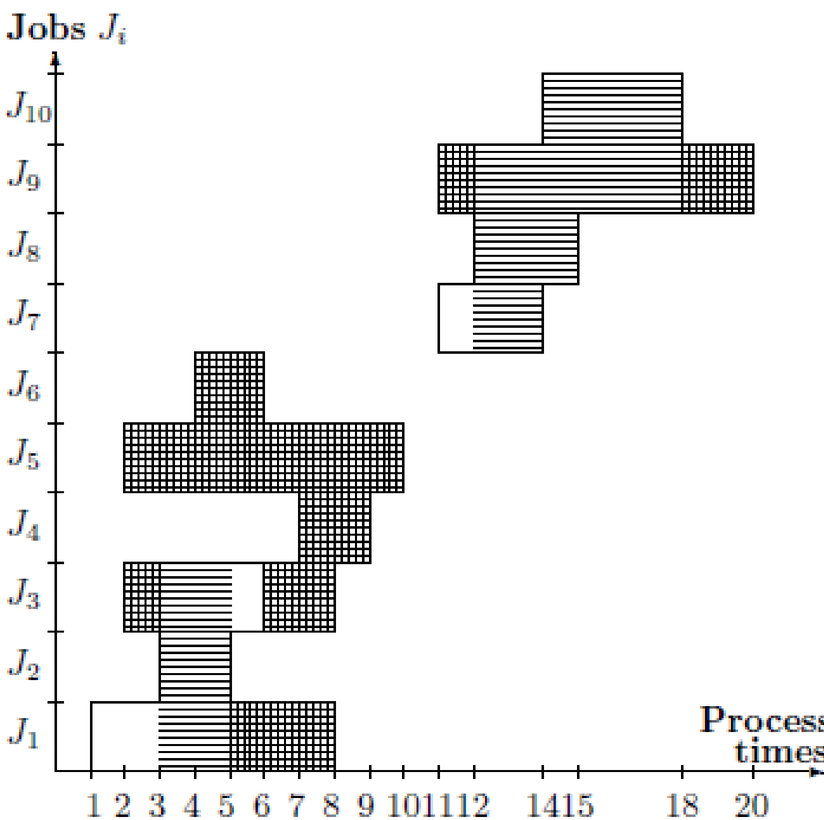
If there does not exist a scenario $p \in T$ such that the permutation π_k is optimal for the instance $1|p|\sum C_i$, then $OR(\pi_k, T) = \varnothing$.

$$OB(\pi_k, T) \subseteq OR(\pi_k, T)$$

The Optimality Region: Example

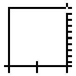
Table 1. Input data for the instance $1|p_i^L \le p_i \le p_i^U|\sum C_i$

i	1	2	3	4	5	6	7	8	9	10
p_i^L	1	3	2	7	2	4	11	12	11	14
p_i^U	8	5	8	9	10	6	14	15	20	18



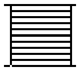
Types of segments for each job $J_{k_r} \in J$ in the fixed permutation

$$\pi_k = (J_{k_1}, J_{k_2}, \ldots, J_{k_n}) \in \Pi$$



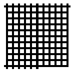
the segment of optimality

$$[l_{k_r}^{opt}, u_{k_r}^{opt}] \subseteq [p_{k_r}^L, p_{k_r}^U]$$



the segment of conditional optimality

$$[l_{k_r}^{copt}, u_{k_r}^{copt}] \subseteq [p_{k_r}^L, p_{k_r}^U]$$



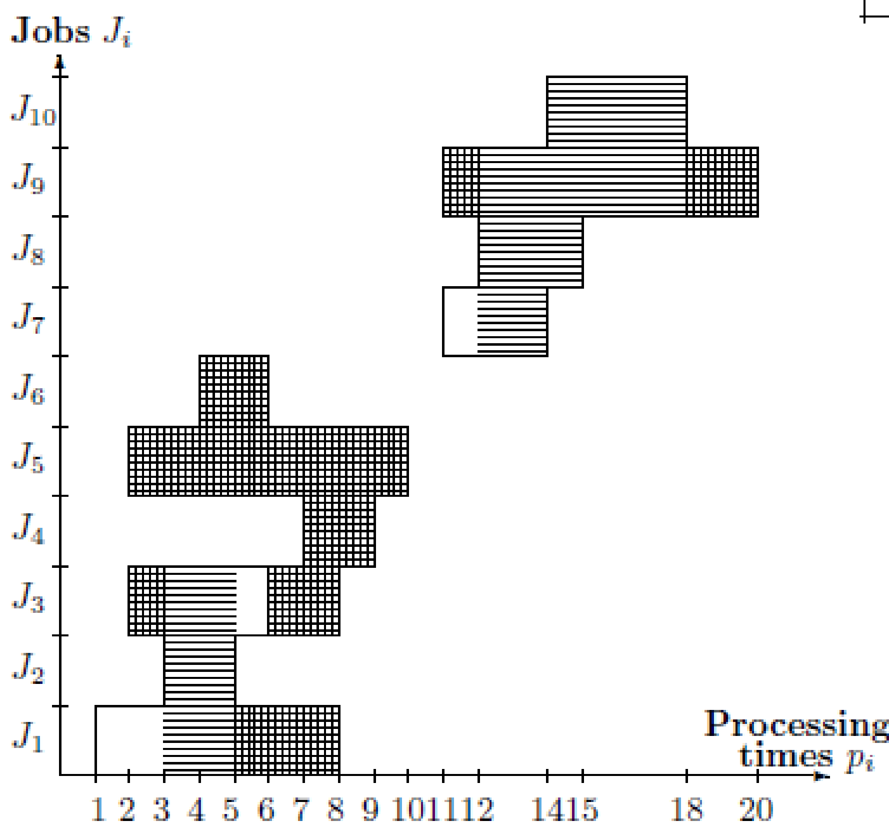
the segment of non-optimality

$$[l_{k_r}^{non}, u_{k_r}^{non}] \subseteq [p_{k_r}^L, p_{k_r}^U]$$

The segment of optimality

Table 1. Input data for the instance $1|p_i^L \leq p_i \leq p_i^U|\sum C_i$

i	1	2	3	4	5	6	7	8	9	10
p_i^L	1	3	2	7	2	4	11	12	11	14
p_i^U	8	5	8	9	10	6	14	15	20	18



Types of segments for each job $J_{k_r} \in J$ in the fixed permutation $\pi_k = (J_{k_1}, J_{k_2}, \ldots, J_{k_n}) \in \Pi$

the segment of optimality $[l_{k_r}^{opt}, u_{k_r}^{opt}] \subseteq [p_{k_r}^L, p_{k_r}^U]$ (Definition 1)

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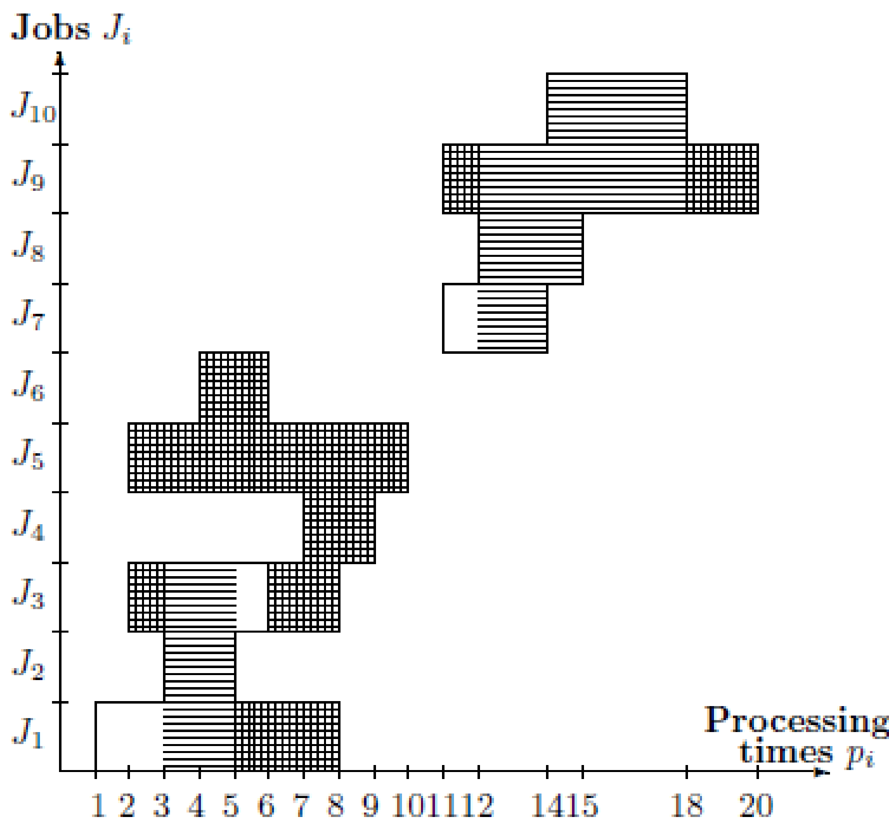
permutation π_k being optimal for the instance $1|p|\sum C_i$ with the scenario $p = (p_1, p_2, \ldots, p_n) \in T$, remains optimal for the instance $1|p'|\sum C_i$ with any scenario

$$p' \in [p_1, p_1] \times [p_2, p_2] \times \ldots \times$$
$$\times [p_{i_g-1}, p_{i_g-1}] \times [l_{i_g}^{opt}, u_{i_g}^{opt}] \times$$
$$\times [p_{i_g+1}, p_{i_g+1}] \times \ldots \times [p_n, p_n].$$

The segment of non-optimality

Table 1. Input data for the instance $1|p_i^L \leq p_i \leq p_i^U|\sum C_i$

i	1	2	3	4	5	6	7	8	9	10
p_i^L	1	3	2	7	2	4	11	12	11	14
p_i^U	8	5	8	9	10	6	14	15	20	18



Types of segments for each job $J_{k_r} \in J$ in the fixed permutation $\pi_k = (J_{k_1}, J_{k_2}, ..., J_{k_n}) \in \Pi$

the segment of non-optimality

$$[l_{k_r}^{non}, u_{k_r}^{non}] \subseteq [p_{k_r}^L, p_{k_r}^U]$$

for any each point $p_{k_r}^* \in [l_{k_r}^{non}, u_{k_r}^{non}]$, the permutation $\pi_k = (J_{k_1}, J_{k_2}, ..., J_{k_n}) \in \Pi$ cannot be optimal for the instance $1|p|\sum C_i$ with any scenario $p = (..., p_{k_r}^*, ...) \in T$

there exists a job $J_{k_v} \in J$, $r < v$,

$$p_{k_v}^U = l_{k_r}^{non} < p_{k_r}^U = u_{k_r}^{non}$$

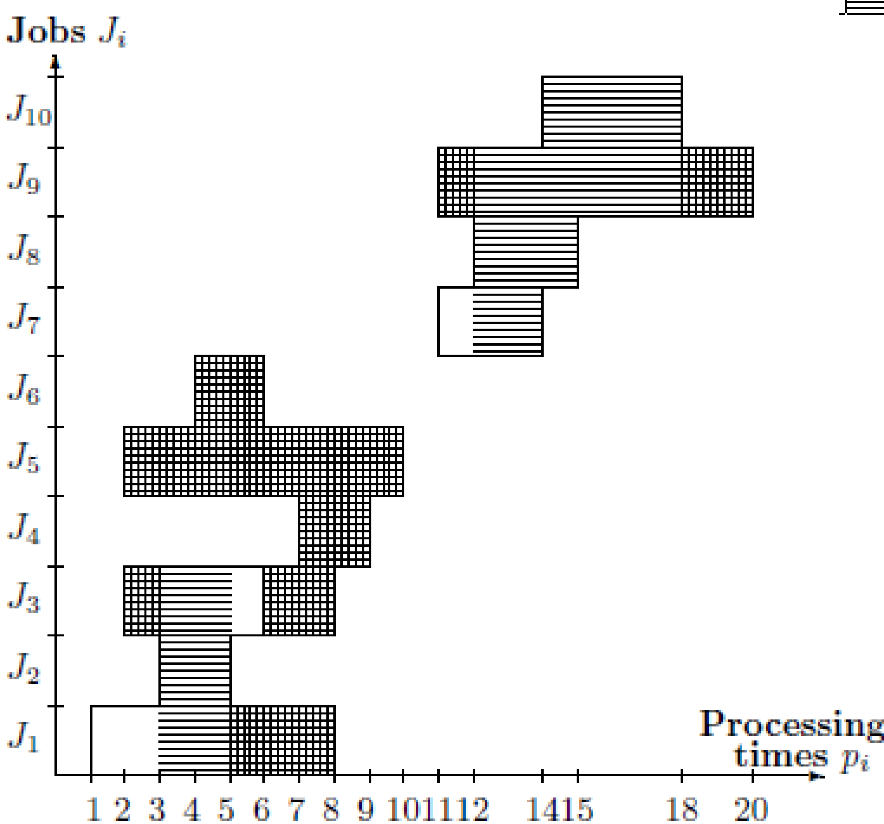
or there exists a job $J_{k_w} \in J$, $w < r$,

$$l_{k_r}^{non} = p_{k_r}^L < u_{k_r}^{non} = p_{k_w}^L$$

The segment of conditional optimality

Table 1. Input data for the instance $1|p_i^L \le p_i \le p_i^U|\sum C_i$

i	1	2	3	4	5	6	7	8	9	10
p_i^L	1	3	2	7	2	4	11	12	11	14
p_i^U	8	5	8	9	10	6	14	15	20	18



Types of segments for each job $J_{k_r} \in J$ in the fixed permutation $\pi_k = (J_{k_1}, J_{k_2}, \dots, J_{k_n}) \in \Pi$

the segment of conditional optimality $[l_{k_r}^{copt}, u_{k_r}^{copt}] \subseteq [p_{k_r}^L, p_{k_r}^U]$

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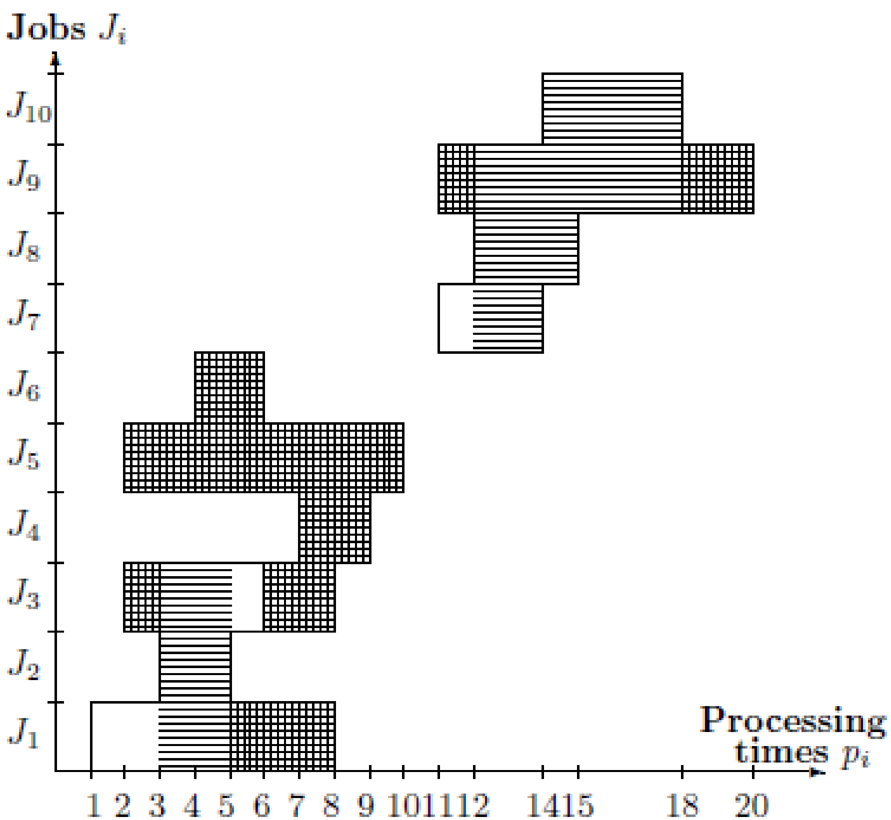
for each point $p_{k_r}^* \in [l_{k_r}^{copt}, u_{k_r}^{copt}]$, $p_{k_r}^* \notin [l_{k_r}^{non}, u_{k_r}^{non}]$, there exists a job $J_{k_d} \in J$, $d \neq r$, with the inclusion $p_{k_r}^* \in [p_{k_d}^L, p_{k_d}^U]$.

$J_{k_r} \in J$

Properties of the segments for the jobs

Table 1. Input data for the instance $1|p_i^L \leq p_i \leq p_i^U|\sum C_i$

i	1	2	3	4	5	6	7	8	9	10
p_i^L	1	3	2	7	2	4	11	12	11	14
p_i^U	8	5	8	9	10	6	14	15	20	18

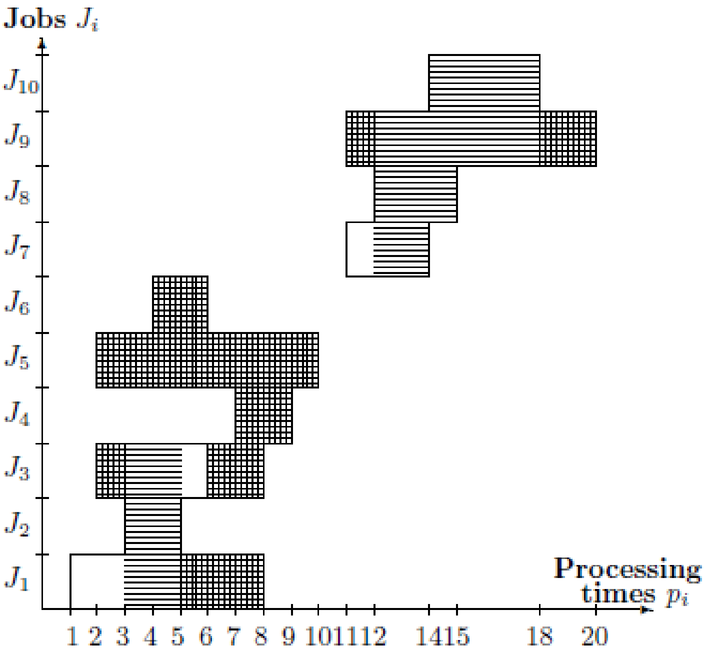


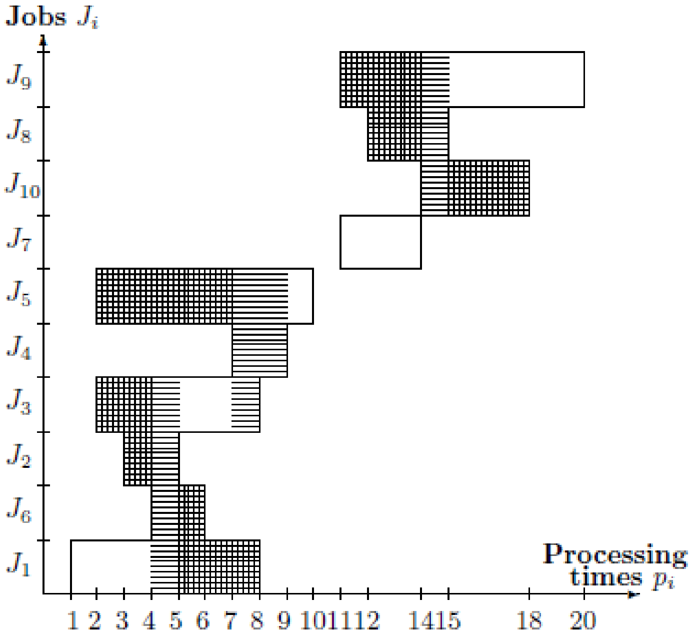
$$(l_{k_r}^{non}, u_{k_r}^{non}) \cap [l_{k_r}^{opt}, u_{k_r}^{opt}] = \varnothing$$

$$[l_{k_r}^{copt}, u_{k_r}^{copt}] \cap (l_{k_r}^{opt}, u_{k_r}^{opt}) = \varnothing$$

$$[l_{k_r}^{copt}, u_{k_r}^{copt}] \cap (l_{k_r}^{non}, u_{k_r}^{non}) = \varnothing$$

Properties of the segments for the jobs





for each job $J_i \in J$ in the permutation $\pi_k \in \Pi$ there may exist

– at most one segment of optimality,

– at most two segments of conditional optimality,

– at most two segments of non-optimality.

Properties of the Optimality Region

Lemma 1. The segment $[p_{k_r}^L, p_{k_r}^U]$ defining all possible durations of the job $J_{k_r} \in J$ is the union of the segment of optimality, the segments of non-optimality and the segments of conditional optimality determined for the job J_{k_r}, $r \in \{1,2,\ldots,n\}$, in the permutation $\pi_k = (J_{k_1}, J_{k_2}, \ldots, J_{k_n}) \in \Pi$.

Theorem 2. The optimality region for the permutation $\pi_k = (J_{k_1}, J_{k_2}, \ldots, J_{k_n}) \in \Pi$ for the instance $1 | p_i^L \le p_i \le p_i^U | \sum C_i$ is equal to the optimality region for the same permutation π_k for the instance $1 | \hat{p}_i^L \le p_i \le \hat{p}_i^U | \sum C_i$ with the reduced segments $[\hat{p}_{k_r}^L, \hat{p}_{k_r}^U]$ of all possible durations of the jobs $J_i \in J$ determined in (5).

$$\hat{p}_{k_r}^L = \max_{1 \le j \le r \le n} \left\{ p_{k_j}^L \right\}, \; \hat{p}_{k_r}^U = \min_{i \le r \le j \le n} \left\{ p_{k_j}^U \right\} \qquad (5)$$

The properties of the Optimality Region

Lemma 2. For the instance $1|\hat{p}_i^L \le p_i \le \hat{p}_i^U|\sum C_i$ with the reduced segments $[\hat{p}_i^L, \hat{p}_i^U]$, $J_i \in J$, of the job durations determined in (5), the segment of optimality $[l_{k_r}^{opt}, u_{k_r}^{opt}]$ for the job $J_{k_r} \in J$ in the permutation $\pi_k \in \Pi$ has no common point with the open interval $(p_{k_d}^L, p_{k_d}^U)$ given for any job $J_{k_d} \in J$, $d \ne r$, i.e., the following equality holds:

$$[l_{k_r}^{opt}, u_{k_r}^{opt}] \cap (p_{k_d}^L, p_{k_d}^U) = \varnothing \tag{6}$$

Theorem 3. Let the strict inequality $p_i^L < p_i^U$ hold for each job $J_i \in J$. The optimality region $OR(\pi_k, T)$ for the permutation $\pi_k = (J_{k_1}, J_{k_2}, ..., J_{k_n}) \in \Pi$ is empty, if and only if there exists at least one job $J_{k_r} \in J$, $r \in \{1,2,...,n\}$, in the permutation $\pi_k = (J_{k_1}, J_{k_2}, ..., J_{k_n}) \in \Pi$, which has no segment of optimality and no conditional optimality.

Algorithms for calculating the sum of the relative optimality sets for the jobs J

Algorithm 1.

– tests the equality $OR(\pi_k, T) = \varnothing$;

– If $OR(\pi_k, T) \neq \varnothing$, then Algorithm 1 constructs an instance $1|\hat{p}_i^L \leq p_i \leq \hat{p}_i^U|\sum C_i$ with the reduced segments $\hat{T}$.

Input: Segments $[p_i^L, p_i^U]$ for the jobs $J_i \in J$; permutation $\pi_k = (J_{k_1}, J_{k_2}, ..., J_{k_n}) \in \Pi$.

Output: Segments $[\hat{p}_i^L, \hat{p}_i^U]$ for the jobs $J_i \in J$ provided that $OR(\pi_k, T) \neq \varnothing$.

Asymptotic complexity:

$O(n)$

Algorithms for calculating the sum of the relative optimality sets for the jobs J

Algorithm 2.
Calculation of the relative perimeter of the optimality region $OR(\pi_k, T)$ for the fixed permutation π_k

Input: A permutation $\pi_k = (J_{k_1}, J_{k_2}, \ldots, J_{k_n})$ such that $OR(\pi_k, T) \neq \varnothing$; segments $[p_i^L, p_i^U]$ and $[\hat{p}_i^L, \hat{p}_i^U]$ of the possible durations of all jobs $J_i \in J$.
Output: The sum $\sum OS(\pi_k)$ of the relative optimality sets $OS(J_{k_r}, \pi_k)$ of jobs $J_{k_r} \in J$ in the permutation π_k.

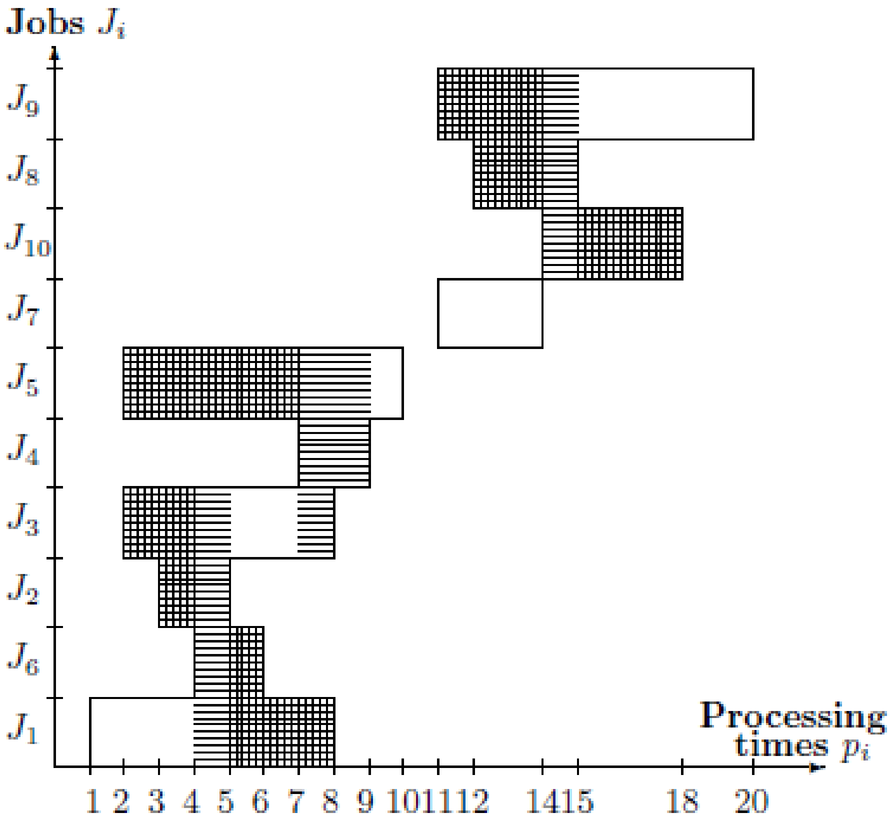
$$\sum OS(\pi_k) = \sum_{r=1}^{n} \frac{OS(J_{k_r}, \pi_k)}{p_{k_r}^U - p_{k_r}^L} \qquad (7)$$

$$OS(J_{k_r}, \pi_k) = (u_{k_r}^{opt} - l_{k_r}^{opt}) + OS_{k_r}^{copt} \qquad (8)$$

$$OS_{k_r}^{copt} = \sum_{j=1}^{n(r)} \frac{u_{k_r}^{copt}(r_j) - l_{k_r}^{copt}(r_j)}{|J_r^j|} \qquad (9)$$

$O(n)$

Calculation of the sum of the relative optimality sets for the jobs J: Example



$$\pi_2 = (J_1, J_6, J_2, J_3, J_4, J_5, J_7, J_{10}, J_8, J_9)$$

$$OS_1^{copt} = OS_6^{copt} = OS_2^{copt} = \frac{5-4}{4} = \frac{1}{4}$$

$$OS_3^{copt} = \frac{5-4}{4} + \frac{8-7}{3} = \frac{7}{12}$$

$$OS_4^{copt} = OS_5^{copt} = \frac{8-7}{3} + \frac{9-8}{2} = \frac{5}{6}$$

$$OS_7^{copt} = 0$$

$$OS_{10}^{copt} = OS_8^{copt} = OS_9^{copt} = \frac{15-14}{3} = \frac{1}{3}$$

$$\sum OS(\pi_2) = \frac{1}{8-1}(3+\frac{1}{4}) + \frac{1}{6-4}(0+\frac{1}{4}) + \frac{1}{5-3}(0+\frac{1}{4}) + \frac{1}{8-2}(2+\frac{7}{12}) + \frac{1}{9-7}(0+\frac{5}{6}) + \frac{1}{10-2}(1+\frac{5}{6}) + \frac{1}{14-11}(3+0) + \frac{1}{18-14}(0+\frac{1}{3}) + \frac{1}{15-12}(0+\frac{1}{3}) + \frac{1}{20-11}(5+\frac{1}{3}) \approx 3.58$$

$$\sum OS(\pi_k) = \sum_{r=1}^{n} \frac{OS(J_{k_r}, \pi_k)}{p_{k_r}^U - p_{k_r}^L}$$

$$OS(J_{k_r}, \pi_k) = (u_{k_r}^{opt} - l_{k_r}^{opt}) + OS_{k_r}^{copt}$$

$$OS_{k_r}^{copt} = \sum_{j=1}^{n(r)} \frac{u_{k_r}^{copt}(r_j) - l_{k_r}^{copt}(r_j)}{|J_r^j|}$$

Computational results

$$\Delta\pi_k = \frac{\sum_{J_i \in J} C_i(\pi_k, p^*) - \sum_{J_i \in J} C_i(\pi_t, p^*)}{\sum_{J_i \in J} C_i(\pi_t, p^*)} \cdot 100\%$$

p^* – factual scenario π_t – actually optimal permutation for the factual scenario

π_b – the permutation having the largest relative perimeter of the optimality box

$\pi_m(\pi_l, \pi_r)$ – the permutation, in which all jobs are ordered according to non-decreasing mid-points (between the left bound and right bound)

n	$\Delta(\pi_b)$	$\Delta(\pi_m)$	$\Delta(\pi_l)$	$\Delta(\pi_r)$	$\sum OS(\pi_b)$	$\sum OS(\pi_m)$	$\sum OS(\pi_l)$	$\sum OS(\pi_r)$
1	2	3	4	5	2	3	4	5
10	1.549414	1.660279	7.424655	6.617099	2.922593	2.911364	2.88077	2.877416
20	0.938677	2.60664	9.722835	6.033818	3.101165	2.928316	2.893201	2.896898
30	1.198182	1.982767	5.827958	5.004542	3.259493	3.056876	2.9568	2.965298
40	1.479872	1.905256	5.572068	3.831046	3.33666	3.135732	2.993827	3.00134
50	1.458717	1.710841	4.776284	3.495722	3.388074	3.189825	3.034753	3.026718
60	1.165505	1.340499	3.372674	3.699239	3.41726	3.212205	3.032864	3.039621
70	0.972522	1.324954	3.70249	2.280066	3.444275	3.244189	3.049236	3.052455
80	1.242643	1.431955	3.337011	2.601677	3.461468	3.266891	3.055648	3.058233
90	0.918133	1.161546	2.616746	2.583409	3.478897	3.285013	3.056571	3.077345
100	0.962029	1.149427	2.597306	2.312391	3.489012	3.289753	3.061477	3.067824
	1.188569	1.627416	4.895003	3.845901	3.329890	3.152016	3.001515	3.006315

Literature

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Thank you for your attention