Multiple crack interaction problem in magnetoelectroelastic solids

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Abstract

The interaction problem of multiple arbitrarily oriented and distributed cracks in magnetoelectroelastic materials is studied. After deriving the elementary solutions for a finite crack under different loading conditions, the present interaction problem is deduced to a system of integral equations with the aid of the pseudo-traction–electric-displacement–magnetic-induction method (abbreviated as PTEDMIM). After the integral equations are solved numerically by using the Chebyshev numerical integration technique, the three modes of stress intensity factors, the electric displacement intensity factor, the magnetic induction intensity factor as well as the mechanical strain energy release rate are evaluated. Detailed comparisons between the results derived under the compound mechanical–electric–magnetic loading conditions and those derived under purely mechanical loading conditions are performed. Some valuable conclusions are given which are certain to be useful for investigating interaction effect among multiple cracks in magnetoelectroelastic solids.

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1. Introduction

Due to possessing both piezoelectric and piezomagnetic features that are vital to electronic device operations, magnetoelectroelastic composites such as the ferrite-ferroelectric composites are attracting more and more attentions and being increasingly used in electronic industry. However, the structural integrity of this kind of materials is still not understood nowadays. They could be vulnerable to mechanical damage when operated under aggressive environments such as high temperature, high stress and/or high moisture. Once cracks are present inside the materials, their structural strength will be degenerated. The device will therefore deteriorate in performance and even fail to work normally. This is undoubtedly undesirable in site. Thus, to investigate the structural integrity and fracture mechanism of such kind of materials has become a problematic task in related research fields.

In order to understand the fracture mechanism of magnetoelectroelastic composite, some preliminary researches have been done in recent years. For instance, Liu et al. (2001) proposed the Green’s functions for anisotropic magnetoelectroelastic solids with an elliptical cavity or a crack; Sih and Song (2003) investigated the magnetic and electric poling effects associated with crack growth in BaTiO$_3$–CoFe$_2$O$_4$ composite; Song and Sih (2003) studied the crack initiation behavior in magnetoelectroelastic composite under in-plane deformation; Spyropoulos et al. (2003) considered the magnetoelectroelastic composite with poling parallel to plane of line crack under out-of-plane deformation; Gao et al. (2003a, 2003b) analyzed the crack problems in magnetoelectroelastic solids; Hou et al. (2003) solved the contact problem of transversely isotropic magnetoelectroelastic bodies. Through foregoing works, a series of significant achievements have been obtained. But compared

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to the related studies in piezoelectric materials (Sosa and Pak, 1990; Sosa, 1991, 1992; Pak, 1990, 1992; Suo et al., 1992; Park and Sun, 1995a, 1995b; Park and Carman, 1997; Tian and Chau, 2003), the research in magnetoelectroelastic composites is still insufficient and further contributions are continually expected. In particular on the research concerning the crack interaction problems, a large number of valuable conclusions have been achieved in piezoelectric materials (Han and Chen, 1999; Chen and Han, 1999a, 1999b; Han and Wang, 1999; Tian and Chen, 2000; Zeng and Rajapakse, 2000). However, to the authors’ knowledge, little effort has been made in magnetoelectroelastic solids.

In view of aforementioned reasons, this paper will deal with the oriented crack interaction problems in magnetoelectroelastic composite. The paper is organized as follows. In Section 2, fundamental solutions for a finite crack subjected to five kinds of loadings are derived. In Section 3, a pseudo-traction–electric-displacement–magnetic-induction method is proposed and the interaction problem is reduced to a system of integral equations that can be solved numerically with the aid of the Chebyshev numerical integration technique. In Section 4, the stress intensity factors, the electric displacement intensity factor, the magnetic induction intensity factor as well as the mechanical strain energy release rate at crack tips are evaluated. Numerical examples and discussions are given in Section 5, and some valuable conclusions are obtained finally in Section 6.

2. Fundamental formula and fundamental solutions

2.1. Fundamental formula

For a linear magnetoelectroelastic medium problem, the elastic, electric and magnetic field equations can be written as (Sih and Song, 2003)

\[
\begin{align*}
\sigma_{ij} &= C_{jks} e_{ks} - e_{ij} E_k - h_{ij} \Delta x, \\
D_i &= e_{iks} e_{ks} + \omega_{ij} E_k + \beta_{ij} \Delta x, \\
B_i &= h_{iks} e_{ks} + \beta_{ij} E_k + \gamma_{ij} \Delta x, \\
e_{ij} &= \frac{1}{2} (u_{ij} + u_{ji}), \\
E_i &= -\varphi_j, \\
\Delta_i &= -\phi_j, \\
\sigma_{ij,j} &= 0, \\
D_{ij} &= 0, \\
B_{i,j} &= 0,
\end{align*}
\]

(1)

where, \(\sigma_{ij}\), \(D_i\), \(B_i\), \(e_{ij}\), \(E_i\), \(\Delta_i\), \(u_i\), \(\varphi\) and \(\phi\) are the stress, electric displacement, magnetic induction, strain, electric field, magnetic field, elastic displacements, electric potential and magnetic potential, respectively; \(\omega_{ij}\) and \(\gamma_{ij}\) represent the dielectric permittivities and magnetic permeabilities, respectively; and \(C_{jks}, e_{sj}, h_{sj}\), and \(\beta_{sj}\) are the elastic, piezoelectric, piezomagnetic and electromagnetic constants, respectively.

It has been shown by Liu et al. (2001) and Gao et al. (2003a), that for a two-dimensional problem, i.e., with geometry and external loading invariant in the direction normal to \(xy\)-plane, the magnetoelectroelastic field can be represented in terms of five functions \(f_1(z_j), f_2(z_j), f_3(z_j), f_4(z_j), \) and \(f_5(z_j)\), each of which is holomorphic in its argument \(z_j = x + \mu_j y\). With these holomorphic functions (or complex potentials), the general solutions for displacements \(u_1, u_2\) and \(u_3\), electric potential \(\varphi\), magnetic potential \(\phi\), stresses \(\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{22}\) and \(\sigma_{23}\), electric displacements \(D_1\) and \(D_2\), and magnetic inductions \(B_1\) and \(B_2\) could be expressed as

\[
\begin{align*}
[u_1] &= 2 \text{Re} \left[ \sum_{j=1}^{5} A_{ij} f_j(z_j) \right], \\
[\sigma_{21}] &= 2 \text{Re} \left[ \sum_{j=1}^{5} L_{ij} f_j'(z_j) \right], \\
[\sigma_{11}] &= -2 \text{Re} \left[ \sum_{j=1}^{5} L_{ij} \mu_j f_j'(z_j) \right],
\end{align*}
\]

(2)

where \([u_1] = [u_1, u_2, u_3, \varphi, \phi]\), \([\sigma_{11}] = [\sigma_{11}, \sigma_{12}, \sigma_{13}, D_1, B_1]\), \([\sigma_{21}] = [\sigma_{21}, \sigma_{22}, \sigma_{23}, D_2, B_2]\), the prime (’ ) denotes differentiation with respect to the associated arguments, and \(A\) and \(L\) are two \(5 \times 5\) matrices depending on material constants. Let \(A_j = [A_{1j}, A_{2j}, A_{3j}, A_{4j}, A_{5j}]\) and \(B_j = [L_{1j}, L_{2j}, L_{3j}, L_{4j}, L_{5j}]\). \(\mu_j\) and \((a_j, b_j)\) are the eigenvalues with positive imaginary parts and the associate eigenvectors of (Liu et al., 2001)

\[\Pi \xi = \mu \xi,\]

(3)

where
\[ \Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_3^T \end{bmatrix}, \quad \xi = \begin{bmatrix} a \\ b \end{bmatrix}, \]  
\[ \Pi_1 = -T^{-1}T^T, \quad \Pi_2 = T^{-1}, \quad \Pi_3 = \Gamma T^{-1}T^T - A, \]  
and  
\[ A = \begin{bmatrix} C_{i1k1} & \epsilon_{i1} & h_{11} \\ \epsilon_{i1k1} & -\omega_{11} & -\beta_{11} \\ h_{11k1} & -\beta_{11} & -\gamma_{11} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} C_{i1k2} & \epsilon_{i2} & h_{21} \\ \epsilon_{i2k2} & -\omega_{12} & -\beta_{12} \\ h_{21k2} & -\beta_{12} & -\gamma_{12} \end{bmatrix}, \quad T = \begin{bmatrix} C_{i2k1} & \epsilon_{21} & h_{22} \\ \epsilon_{21k2} & -\omega_{22} & -\beta_{22} \\ h_{22k2} & -\beta_{22} & -\gamma_{22} \end{bmatrix}. \]  

As done by Suo (1990), we introduce one function vector \( f(z) \)
\[ f(z) = [f_1(z), f_2(z), f_3(z), f_4(z), f_5(z)]^T. \]  
Another solution form appropriate for the method of analytic continuation is written as
\[ \mathbf{u} = 2 \text{Re}[\mathbf{Af}(z)], \]  
\[ [\sigma_{12}] = 2 \text{Re}[\mathbf{L}f'(z)], \]  
\[ [\sigma_{12}] = 2 \text{Re}[\mathbf{W}f'(z)]. \]  
where \( \mathbf{W}_j = [-L_1\mu_j, -L_2\mu_j, -L_3\mu_j, -L_4\mu_j, -L_5\mu_j] \), the argument has the generic form \( z = x + i\mu y \). Once the solution of \( f(z) \) is obtained for a given boundary value problem, a replacement of \( z_1, z_2, z_3, z_4 \) and \( z_5 \) should be made for each component function to calculate field quantities from Eq. (2).

Moreover, define
\[ \mathbf{H} = 2 \text{Re}(i\mathbf{AL}^{-1}). \]  
Here, \( i = \sqrt{-1}. \)

Consider an in-plane coordinate rotation (Suo, 1990; Suo et al., 1992)
\[ [V_{ij}] = \begin{bmatrix} \frac{\partial x^*}{\partial x_j} \\ \frac{\partial y^*}{\partial x_j} \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \]  
where (*) indicates the new coordinate system. After the quantities \( \mu_j, \mathbf{A}, \mathbf{L} \) and \( \mathbf{H} \) in one coordinate system are calculated out, the corresponding quantities in new coordinate system can be obtained following the following transformations
\[ \mu_j^* = (\mu_j \cos \theta - \sin \theta)/(\mu_j \sin \theta + \cos \theta), \]  
\[ \mathbf{A}^* = \mathbf{VA}, \quad \mathbf{L}^* = \mathbf{VL}, \quad \mathbf{H}^* = \mathbf{VHV}^T. \]

2.2. Fundamental solutions for a finite crack

As shown in Fig. 1, consider an infinite magneto-electroelastic plate contains a finite crack of length \( 2a \) subjected to concentrated tractions \( P, Q \) and \( R \), concentrated electric displacement \( D_2 \) and concentrated magnetic induction \( B_2 \), respectively. Assume there exist three coordinate systems. They are a global system denoted by \( x_1y_1 \), and two local systems respectively denoted by \( xy \) and \( x_2y_2 \). The origin of the local system \( x_2y_2 \) is located at \( z_0(x_0, y_0) \), which is the center of the crack. The \( x \)-axis is along the crack line, while the \( y \)-axis is perpendicular to it. The origin of the local system \( x_2y_2 \) is located at the point \( z_0^*(x_0^*, y_0^*) \). \( \theta \) represents the included angle between \( x \)-axis and \( x_2 \)-axis and \( \theta^* \) the included angle between \( x_2 \)-axis and \( x_1 \)-axis. The magnetic and electric poling directions of the material are assumed to be along the \( y_1 \)-axis.

According to the continuity of the tractions, the electric displacement and the magnetic induction across the whole \( x \)-axis, the following relation is obtained
\[ \left[ \sigma_{22}(x) \right]^+ = \left[ \sigma_{22}(x) \right]^-. \]  
from Eq. (9) we have
\[ \left[ \mathbf{L}f'(x) - \mathbf{L}f'(x) \right]^+ = \left[ \mathbf{L}f'(x) - \mathbf{L}f'(x) \right]^- \]  
(15)
Fig. 1. A finite crack under different loading conditions. (a) Under concentrated traction $P$, (b) under concentrated traction $Q$, (c) under concentrated traction $R$, (d) under concentrated electric displacement $D_2$, (e) under concentrated magnetic induction $B_2$.

which is the simplest Riemann–Hilbert problem whose solution is holomorphic function. Considering the remote conditions, the holomorphic function should be zero and then the following relation can be derived

$$L f'(z) = \overline{L f'(z)}.$$  \hfill (16)

The boundary condition on the crack faces as shown in Fig. 1 is

$$[L f'(x)]^+ + [\overline{L f'(x)}]^- = \delta(x - s) \quad (|x| < a),$$  \hfill (17)

where $s$ denotes the distance between the acting point of loading and the origin point $z_0$, $t$ represents the concentrated traction or electric displacement or magnetic induction prescribed on the crack faces.

With the aid of Eq. (16), have

$$[L f'(x)]^+ + [L f'(x)]^- = \delta(x - s) \quad (|x| < a).$$  \hfill (18)
Introduce

$$\Omega(z) = Lf'(z).$$

Eq. (18) is then rewritten as

$$\Omega^+(s) + \Omega^-(s) = \mathbf{b}(s-a) \quad (|s| < a).$$

Eq. (20) is a typical Riemann–Hilbert problem. We can derive the solution for the problem as follows

$$\Omega(z) = \frac{t}{2\pi(s-z)} \left( \frac{a^2 - s^2}{z^2 - a^2} \right)^{1/2}.$$  \hspace{1cm} (21)

According to Eqs. (19) and (21), have

$$f'(z) = L^{-1} \left[ \frac{t}{2\pi(s-z)} \left( \frac{a^2 - s^2}{z^2 - a^2} \right)^{1/2} \right].$$  \hspace{1cm} (22)

For the crack under concentrated traction \( P \) shown in Fig. 1(a), \( t = [0, P, 0, 0, 0] \). Then, the stresses \( g_{nnp}, g_{ntp}^1 \) and \( g_{ntp}^2 \), the electric displacement \( g_{2Dp} \) and the magnetic induction \( g_{2Bp} \) at point \( z^* \) due to the concentrated traction \( P \) can be calculated by Eqs. (22) and (2), i.e.,

$$s_{nnp}(s, \theta, z_o, s^*, \theta^*, z^*_o) = 2 \text{Re} \sum_{j=1}^{5} \left[ -L_{1j} j \mu_j \sin^2(\theta^* - \theta) + L_{2j} j \cos^2(\theta^* - \theta) - L_{1j} j \sin(\theta^* - \theta) \right] f_j'(z_j),$$

$$s_{ntp}^1(s, \theta, z_o, s^*, \theta^*, z^*_o) = \text{Re} \sum_{j=1}^{5} \left[ L_{1j} j \sin(\theta^* - \theta) + L_{2j} j \sin(\theta^* - \theta) + 2L_{1j} j \sin(\theta^* - \theta) \right] f_j'(z_j),$$

$$s_{ntp}^2(s, \theta, z_o, s^*, \theta^*, z^*_o) = 2 \text{Re} \sum_{j=1}^{5} \left[ L_{3j} j \sin(\theta^* - \theta) + L_{4j} j \cos(\theta^* - \theta) \right] f_j'(z_j),$$

$$g_{2Dp}(s, \theta, z_o, s^*, \theta^*, z^*_o) = 2 \text{Re} \sum_{j=1}^{5} \left[ L_{4j} j \sin(\theta^* - \theta) + L_{4j} j \cos(\theta^* - \theta) \right] f_j'(z_j),$$

$$g_{2Bp}(s, \theta, z_o, s^*, \theta^*, z^*_o) = 2 \text{Re} \sum_{j=1}^{5} \left[ L_{5j} j \sin(\theta^* - \theta) + L_{5j} j \cos(\theta^* - \theta) \right] f_j'(z_j),$$

where \( z_j = \text{Re}[s^* e^{i(\theta^*-\theta)} + (z^*_o-z_o) e^{-i\theta}] + \mu_j \text{Im}[s^* e^{i(\theta^*-\theta)} + (z^*_o-z_o) e^{-i\theta}], z_o = x_o + iy_o, z^*_o = x_o^* + iy_o^*, i = \sqrt{-1} \) and \( s^* \) represents the distance from point \( z^*_o \) to the origin point \( z_o \) (see Fig. 1).

Likewise, the stresses \( s_{nnq}, s_{ntq}^1 \) and \( s_{ntq}^2 \), the electric displacement \( g_{2Dq} \) and the magnetic induction \( g_{2Bq} \) at point \( z^* \) due to the concentrated traction \( Q \) (\( t = [0, 0, 0, 0, 0] \)), the stresses \( g_{nnr}, g_{ntr}^1 \) and \( g_{ntr}^2 \), the electric displacement \( g_{2Dr} \) and the magnetic induction \( g_{2Br} \) at point \( z^* \) due to the concentrated traction \( R \) (\( t = [0, 0, R, 0, 0] \)), the stresses \( g_{nnb}, g_{ntb}^1 \) and \( g_{ntb}^2 \), the electric displacement \( g_{2Db} \) and the magnetic induction \( g_{2Bb} \) at point \( z^* \) due to the concentrated electric displacement \( D_2 \) (\( t = [0, 0, 0, D_2, 0] \)), and the stresses \( g_{nnh}, g_{nhh}^1 \) and \( g_{nhh}^2 \), the electric displacement \( g_{2Dh} \) and the magnetic induction \( g_{2Bh} \) at point \( z^* \) due to the concentrated magnetic induction \( B_2 \) (\( t = [0, 0, 0, 0, B_2] \)) can also be calculated with the aid of Eqs. (22) and (2), their expressions are not listed any longer for attaining a concise context of the paper.


Consider an infinite magneto-electroelastic plane containing \( N \) arbitrarily distributed cracks, as shown in Fig. 2(a). Here, the remote loadings are composed of the remote stresses \( \sigma^{12}_{xx}, \sigma^{11}_{xx} \) and \( \sigma^{12}_{xx} \), the remote electric displacements \( D_{11}^{\infty} \) and \( D_{21}^{\infty} \) and the remote magnetic inductions \( B_{11}^{\infty} \) and \( B_{21}^{\infty} \). The magnetic and electric poling directions of the material are along the \( y \)-axis of the global system \( \chi \). The local Cartesian systems \( x_k y_k (k = 1, 2, \ldots, N) \) situated at the center \( o_k \) of each crack is employed with \( x_k \)-axis coincide with the crack surface. \( 2a_k \) and \( \theta_k \) indicate the length and the oriented angle of the \( k \)th crack. All crack faces are assumed to be free of force, external charge and electric current. As done by Horii and Nemat-Nasser (1985) and Horii and Nemat-Nasser (1987) in elastic medium and Han and Chen (1999) in piezoelectric medium, the original interaction problem shown in Fig. 2(a) could be divided into \( N + 1 \) subproblems (see Fig. 2 (b) and (c)).
In subproblem \( N + 1 \) (Fig. 2(b)), the magnetoelectroelastic solid is loaded by the remote stresses \( \sigma_{12}^\infty, \sigma_{11}^\infty \) and \( \sigma_{12}^\infty \), the remote electric displacement \( D_1^\infty \) and \( D_2^\infty \) and the remote magnetic inductions \( B_1^\infty \) and \( B_2^\infty \), but without any crack contained. Subproblem \( k \) \((k = 1, 2, \ldots, N)\) only contains the \( k \)th crack with unknown tractions \( P_k(t_k), Q_k(t_k) \) and \( R_k(t_k) \), unknown electric displacement \( D_k(t_k) \) and unknown magnetic induction \( B_k(t_k) \) attached on both faces (see Fig. 2(c)). Using the fundamental solutions derived above and the superimposing technique, the present crack interaction problem can be reduced to a system of Fredholm integral equations as follows

\[
P_k(s_k) + \sum_{i=1}^{N} \int_{s_i \neq s_k} a_i \left[ g_{nnp}(s_i, \theta_i, \alpha_i, s_k, \theta_k, \alpha_k) P_i(s_i) + g_{nnq}(s_i, \theta_i, \alpha_i, s_k, \theta_k, \alpha_k) Q_i(s_i) + g_{nnr}(s_i, \theta_i, \alpha_i, s_k, \theta_k, \alpha_k) R_i(s_i) + g_{nd}(s_i, \theta_i, \alpha_i, s_k, \theta_k, \alpha_k) D_i(s_i) + g_{nb}(s_i, \theta_i, \alpha_i, s_k, \theta_k, \alpha_k) B_i(s_i) \right] ds_i = p_k(s_k), \tag{24a}
\]
\[
Q_k(s_k) + \sum_{i=1}^{N} \int \left[ g_{ntq}^{i}(s_i, \theta_i, o_i, s_k, \theta_k, o_k) P_i(s_i) + g_{ntq}^{i}(s_i, \theta_i, o_i, s_k, \theta_k, o_k) Q_i(s_i) \right] ds_i = q_k(s_k),
\]
\[
R_k(s_k) + \sum_{i=1}^{N} \int \left[ g_{ntq}^{2}(s_i, \theta_i, o_i, s_k, \theta_k, o_k) P_i(s_i) + g_{ntq}^{2}(s_i, \theta_i, o_i, s_k, \theta_k, o_k) Q_i(s_i) \right] ds_i = r_k(s_k),
\]
\[
D_k(s_k) + \sum_{i=1}^{N} \int \left[ g_{ntb}^{i}(s_i, \theta_i, o_i, s_k, \theta_k, o_k) P_i(s_i) + g_{ntb}^{i}(s_i, \theta_i, o_i, s_k, \theta_k, o_k) Q_i(s_i) \right] ds_i = d_{2k}(s_k),
\]
\[
B_k(s_k) + \sum_{i=1}^{N} \int \left[ g_{ntb}^{2}(s_i, \theta_i, o_i, s_k, \theta_k, o_k) P_i(s_i) + g_{ntb}^{2}(s_i, \theta_i, o_i, s_k, \theta_k, o_k) Q_i(s_i) \right] ds_i = b_{2k}(s_k),
\]
\[ K^R_I = -\left( \pi a \right)^{-1/2} \int_{-a_k}^{a_k} \left( \frac{a_k + s_k}{a_k - s_k} \right)^{1/2} P_k(s_k) \, ds_k, \]
\[ K^R_{II} = -\left( \pi a \right)^{-1/2} \int_{-a_k}^{a_k} \left( \frac{a_k + s_k}{a_k - s_k} \right)^{1/2} Q_k(s_k) \, ds_k, \]
\[ K^R_{III} = -\left( \pi a \right)^{-1/2} \int_{-a_k}^{a_k} \left( \frac{a_k + s_k}{a_k - s_k} \right)^{1/2} R_k(s_k) \, ds_k, \]
\[ K^e_I = -\left( \pi a \right)^{-1/2} \int_{-a_k}^{a_k} \left( \frac{a_k + s_k}{a_k - s_k} \right)^{1/2} D_k(s_k) \, ds_k, \]
\[ K^m_I = -\left( \pi a \right)^{-1/2} \int_{-a_k}^{a_k} \left( \frac{a_k + s_k}{a_k - s_k} \right)^{1/2} B_k(s_k) \, ds_k, \]

for the right crack tip, and
\[ K^L_I = -\left( \pi a \right)^{-1/2} \int_{-a_k}^{a_k} \left( \frac{a_k - s_k}{a_k + s_k} \right)^{1/2} P_k(s_k) \, ds_k, \]
\[ K^L_{II} = -\left( \pi a \right)^{-1/2} \int_{-a_k}^{a_k} \left( \frac{a_k - s_k}{a_k + s_k} \right)^{1/2} Q_k(s_k) \, ds_k, \]
\[ K^L_{III} = -\left( \pi a \right)^{-1/2} \int_{-a_k}^{a_k} \left( \frac{a_k - s_k}{a_k + s_k} \right)^{1/2} R_k(s_k) \, ds_k, \]
\[ K^e_L = -\left( \pi a \right)^{-1/2} \int_{-a_k}^{a_k} \left( \frac{a_k - s_k}{a_k + s_k} \right)^{1/2} D_k(s_k) \, ds_k, \]
\[ K^m_L = -\left( \pi a \right)^{-1/2} \int_{-a_k}^{a_k} \left( \frac{a_k - s_k}{a_k + s_k} \right)^{1/2} B_k(s_k) \, ds_k, \]

for the left crack tip, where subscripts I, II and III respectively indicate the Mode I, Mode II and Mode III stress intensity factors, \( e \) the electric displacement intensity factor and \( m \) the magnetic induction intensity factor.

The Mode I and Mode II mechanical strain energy release rates (MSERR) are
\[ G^M_I = \lim_{\delta \to 0} \frac{1}{2 \delta} \int_0^\delta \sigma_{22}(x) \Delta u_2(\delta - x) \, dx \quad \text{for the Mode I} \]  
\[ G^M_{II} = \lim_{\delta \to 0} \frac{1}{2 \delta} \int_0^\delta \sigma_{12}(x) \Delta u_1(\delta - x) \, dx \quad \text{for the Mode II}, \]

where the superscript \( M \) refers to the MSERR, \( \Delta u_1 \) and \( \Delta u_2 \) are the jumps of the displacement components measured from the lower face to the upper face of the crack.

Both the Mode I and Mode II MSERRs are related to the crack tip SIFs, EDIF and MIIF, i.e.,
\[ G^M_I = \frac{1}{4} \left[ H_{21} K_I K_{II} + H_{22} (K_I)^2 + H_{23} K_I K_{III} + H_{24} K_I K_e + H_{25} K_I K_m \right], \]

and
\[ G^M_{II} = \frac{1}{4} \left[ H_{11} (K_{II})^2 + H_{12} K_I K_{II} + H_{13} K_{II} K_{III} + H_{14} K_{II} K_e + H_{15} K_{II} K_m \right]. \]
5. Numerical examples and discussion

Consider an infinite plate made of magnetoelectroelastic material BaTiO$_3$–CoFe$_2$O$_4$, which contains two cracks AB and CD with the same length $2a$, as shown in Fig. 3. Where, $\theta$ and $\psi$ respectively represent the orientation and location angles of crack CD, $r$ the distance between the crack tip A and the center point $o_2$ of crack CD, $d$ the distance between two center points $o_1$ and $o_2$. Effective material constants of the material BaTiO$_3$–CoFe$_2$O$_4$ are listed in Table 1 (Song and Sih, 2003; Li, 2000). Suppose the magnetic and electric poling directions are perpendicular to the parallel crack AB and the plane strain condition is considered.

In order to verify the correctness of the proposed strategy for solving the present crack interaction problem, a comparison of the computational results obtained by using the proposed method and those having been derived in ideal piezoelectric material was made at first. Consider a fictitious magnetoelectroelastic material with the same piezoelectric and elastic constants as PZT-4 and identical magnetic permeabilities as BaTiO$_3$–CoFe$_2$O$_4$ as well as negligible piezomagnetic constants. Three kinds of remote loading conditions are considered, i.e., (i) $\sigma_{\infty 11} = \sigma_{\infty 12} = 0$, $\sigma_{\infty 22} \neq 0$, $D_{\infty 2} = 10^{-8} \sigma_{\infty 22}$ CN$^{-1}$, $B_{\infty 2} = 0$, (ii) $\sigma_{\infty 11} = \sigma_{\infty 12} = 0$, $\sigma_{\infty 22} \neq 0$, $D_{\infty 2} = -10^{-8} \sigma_{\infty 22}$ CN$^{-1}$, $B_{\infty 2} = 0$, and (iii) $\sigma_{\infty 11} = \sigma_{\infty 12} = 0$, $\sigma_{\infty 22} \neq 0$, $D_{\infty 2} = 0$, $B_{\infty 2} = 0$. The normalized Mode I SIF $K_I^A / K_I^\infty (K_I^\infty = \sqrt{\pi a} \sigma_{\infty 22})$ at crack tip A against the location angle $\psi$ is shown in Fig. 4. Clearly seen that the results shown in Fig. 4 agree well with those given by Han and Chen (1999).

In the following, the crack interaction behaviors in magnetoelectroelastic materials are investigated in detail. The normalized Mode I SIF, the normalized MIIF and the normalized Mode I MSERR for various crack geometries are evaluated under different combined mechanical–electric–magnetic loading conditions. Necessary to note that in following calculations, identical
The normalized stress intensity factor $K_1^A/K_1^∞$ versus the angle $ψ$. 

From Fig. 5 (a) and (b), it can be found that the mechanical loading and the magnetic loading are coupled together to take effect on the crack interaction behaviors. Moreover, similar as the positive and negative electric loadings play in piezoelectric materials (Han and Chen, 1999), the positive and the negative magnetic loadings take an opposite effect on the $K_1^A/K_1^∞$ in magnetoelectroelastic materials. For instance, in the ranges of angle $ψ$ from 7.1 deg to 59.6 deg for $r/a = 1.2$ and from 11.9 deg to 63.9 deg for $r/a = 1.5$ (see Fig. 5(b)), the positive magnetic loading increases the $K_1^A/K_1^∞$, while the negative magnetic loading decreases it. Inversely, in the ranges of angle $ψ$ from 59.6 deg to 150.7 deg for $r/a = 1.2$ and from 63.9 deg to 141.3 deg for $r/a = 1.5$, the positive magnetic loading leads to the decrease of the $K_1^A/K_1^∞$, while the negative magnetic loading results in the increase of it. The opposite influence of the positive and negative magnetic loadings on the $K_1^A/K_1^∞$ can be also found in the ranges of angle $ψ$ from 0 deg to 7.1 deg and from 150.7 deg to 170 deg for $r/a = 1.2$, and from 0 deg to 11.9 deg and from 141.3 deg to 170 deg for $r/a = 1.5$. To be the most interested is that three special location angles appear in both Fig. 5 (a) and (b) at which neither the positive magnetic loading nor the negative magnetic loading takes effect on the $K_1^A/K_1^∞$, so that the curves derived under different loading conditions intersect with each other at these angles. Such kind of angles could be called the neutral magnetic loading angle (NMLA) denoted by $ψ_{NM}$. It is observed that $ψ_{NM} = 11.9, 63.9$ and $ψ_{NM} = 11.9, 63.9$ for $r/a = 1.2$, and $ψ_{NM} = 11.9, 63.9$ and $ψ_{NM} = 11.9, 63.9$ for $r/a = 1.5$, respectively. In addition, it is noticed that the positive magnetic loading tends to distend the amplification region ($K_1^A/K_1^∞ > 1$) and deflate the shielding region ($K_1^A/K_1^∞ < 1$); while the negative magnetic loading tends to deflate the amplification region($K_1^A/K_1^∞ > 1$) and distend the shielding region($K_1^A/K_1^∞ < 1$). Furthermore, the positive magnetic loading increases the maximum amplification effect and the maximum shielding effect, while the negative magnetic loading decreases them. It is worth pointing out that, as found by Gao et al. (2003b) in the means of theoretical analysis, the numerical results illustrated in Fig. 5 (a) and (b) further confirm that the stress intensity factor is actually independent of the magnetic loading in collinear crack case (i.e., $ψ = 0$ deg). Besides, the comparison of Fig. 5 (a) and (b) discloses that the interaction between cracks is weakened when the distance $r/a$ increases from 1.2 to 1.5 whatever in the amplification region or in the shielding region.

Subsequently, take the electric loading into account and let the remote electric loading $D_{2e}^∞ = 10^{-8}σ_{22}^∞$ CN$^{-1}$ the normalized Mode I SIF $K_1^A/K_1^∞$ at crack tip A is calculated as a function of the location angle $ψ$ under different remote magnetic induction loading $B_{2e}^∞$. The numerical results are plotted in Fig. 5 (c) and (d). From Fig. 5 (c) and (d), the similar phenomena can also be observed except that the participation of the positive electric loading further enhances the crack interaction effect in both the amplification region and the shielding region, and shifts the maximum amplification and the maximum shielding angles. This fully proves that the mechanical, electric and magnetic loadings are actually coupled together to take effect on the crack interaction effect. After considering the electric loading, the positive and negative magnetic loadings still take an opposite effect on the crack interactions, for example, the positive magnetic loading increases the maximum amplification and the maximum shielding effects while the negative magnetic loading decreases them. The comparisons of Fig. 5 (a) and (c) and Figs. 5(b) and 5(d) disclose that the electric loading does not change the neutral magnetic loading angles.
though it brings a significant influence on the crack interaction effect in magnitude and shifts both the maximum amplification angle and the maximum shielding angle. In other words, identical neutral magnetic loading angles can be found under different electric loading conditions, i.e., \( \psi_{NM} = 7.1, 59.6 \) and 150.7 deg for \( r/a = 1.2 \), and \( \psi_{NM} = 11.9, 63.9 \) and 141.3 deg for \( r/a = 1.5 \), respectively. The independence of the stress intensity factor on magnetic loading in collinear crack case can be found once again from the results shown in Fig. 5 (c) and (d).

In order to investigate the influence of the existence of magnetic loading on the crack interaction behaviors in magnetoelectroelastic material in a more explicit means, the following calculation was further performed. Take the remote magnetic loading \( B_\infty = 0 \) and \( B_\infty = 10^{-6} \sqrt{\pi A^{-1}} \) m, respectively. The normalized Mode I SIF \( K_I^A/K_\infty \) at crack tip \( A \) is calculated as a function of the location angle \( \psi \) under different remote electric displacement loading \( D_\infty \). The results are plotted in Fig. 6 (a) and (b). Similar as found by Han and Chen (1999) in piezoelectric material, Fig. 6 (a) and (b) show that the positive and negative electric loadings take an opposite effect on the \( K_I^A/K_\infty \) in magnetoelectroelastic material regardless of whether the magnetic loading has been applied. By comparing Fig. 6 (a) and (b), it is found that the participation of the positive magnetic loading further enhances the crack interaction effect whatever in the amplification region \( (K_I^A/K_\infty > 1) \) or in the shielding region \( (K_I^A/K_\infty < 1) \), but does not change the neutral electric loading angle (NELA) \( \psi_{NE} \) at which neither the positive nor the negative electric loading creates effect on the \( K_I^A/K_\infty \). Despite whether the magnetic loading is applied or not, identical neutral electric loading angles can be observed from Fig. 6 (a) and (b). They are \( \psi_{NE} = 8.72, 61.2 \) and 149.3 deg. In addition, it is similarly noticed that the electric loading does not bring any influence on the \( K_I^A/K_\infty \) in collinear crack case.

The magnetic induction intensity factors \( K_m^A/K_m^\infty \) \( (K_m^\infty = 10^{-6} \sqrt{\pi A^{-1}} \) m) (MIIF) at the crack tip \( A \) against the location angle \( \psi \) are plotted in Fig. 7 (a) and (b) respectively for \( r/a = 1.2 \) and 1.5. It is found that the positive and negative magnetic loadings take an opposite effect on the MIIF, i.e., in all crack geometric configurations, the positive magnetic loading...
always increases the MIIF, while the negative magnetic loading always decreases it. In addition, it is seen that the $K_{Am}/K_{m}^{\infty}$ is affected sensitively by the angle $\psi$ mainly when $\psi < 50 \text{ deg}$. The influence tends to be larger and larger with decreasing $\psi$ and the maximum influence occurs at the collinear crack situation. When the angle $\psi$ is larger than 50 deg, the influence of $\psi$ on the MIIF becomes very small so that it even could be neglected completely.

The normalized Mode I mechanical strain energy release rate $G_{MA}^{I}/G_{M}^{\infty}$ against the location angle $\psi$ are also calculated under different loading conditions. The computed results are shown in Fig. 8. From Fig. 8, it can be found that, in spite of the distance $r/a$ and whatever whether the electric loading has been applied or not, the positive magnetic loading generally increases the $G_{MA}^{I}/G_{M}^{\infty}$ while the negative magnetic loading decreases it. This suggests that, if take the Mode I mechanical strain energy release rate $G_{MA}^{I}/G_{M}^{\infty}$ as a measure for fracture evaluation, the crack subjected to positive magnetic loading will be more inclined to propagate than that subjected to negative magnetic loading. Moreover, the participation of the positive electric loading further deteriorates this situation. This conclusion is supported by the much larger magnitudes of the $G_{MA}^{I}/G_{M}^{\infty}$ shown in Fig. 8 (c) and (d) than those in Fig. 8 (a) and (b). However, it is noticed from Fig. 8 that in collinear crack case, the $G_{MA}^{I}/G_{M}^{\infty}$ is not independent of the magnetic loading. This is distinctly different from that found from Figs. 5 and 6. Therefore, further practical experiments are still expected to discover the truth.

Furthermore, the normalized Mode I stress intensity factor $K_{MA}^{I}/K_{M}^{\infty}$ as the function of the orientation angle $\theta$ is calculated.

The results derived under different remote loading conditions are as shown in Fig. 9. From Fig. 9, it is found that, despite whether the electric loading is applied or not, there always exist four identical neutral magnetic loading angles occur when $\psi = 0 \text{ deg}$
and two identical neutral magnetic loading angles occur when $\psi = 45$ deg. These angles are $\theta_{NM} = 10.5$, 74.1, 105.9 and 169.5 deg for $\psi = 0$ deg, $\theta_{NM} = 46.2$ and 105.8 deg for $\psi = 45$ deg, respectively. Moreover, as found above, the positive and negative magnetic loadings always take an opposite effect on the $K_I^A/K_I^\infty$ in the whole range of the orientation angle $\theta$. In addition, the different variable tendencies of the curves derived respectively when $\psi = 0$ and 45 deg show that the crack geometric configuration has a great influence on the $K_I^A/K_I^\infty$. In particular when the angle $\psi = 45$ deg, the magnitude of the $K_I^A/K_I^\infty$ is more sensitive to the variation of the electric loading (see Fig. 9 (b) and (d)). This implies that the stability of the crack residing in magnetoelectroelastic material is not only dependent on the remote loading condition, but also related to the geometric configuration of the cracks, for example, the location and the orientation of the crack.

The magnetic induction intensity factors $K_I^{mA}/K_I^{m\infty}$ (MIIF) at the crack tip $A$ against the orientation angle $\theta$ are plotted in Fig. 10 (a) and (b), where $\psi$ is taken to be 0 and 45 deg, respectively. From Fig. 10, the opposite effect of the positive and negative magnetic loadings on the MIIF is also found, i.e., in the whole range of the orientation angle $\theta$, the positive magnetic loading always increases the MIIF, while the negative magnetic loading always decreases it. Moreover, it is noticed that the maximum influence of the magnetic loading on MIIF occurs at $\theta = 0$ deg when $\psi = 0$ deg, while the maximum influence occurs at $\theta = 52$ deg when $\psi = 45$ deg.

Under the corresponding compound loading conditions, the Mode I mechanical strain energy release rate $G_I^{MA}/G_I^{M\infty}$ is also computed as the function of the orientation angle $\theta$. The calculation results are shown in Fig. 11. From Fig. 11, it is noticed that the neutral magnetic loading angles occur in Fig. 9 disappear from the curves of the $G_I^{MA}/G_I^{M\infty}$. But the opposite effect of the positive and negative magnetic loadings on the $G_I^{MA}/G_I^{M\infty}$ and the enhancement function of the positive electric loading on the $G_I^{MA}/G_I^{M\infty}$ can be also found. Moreover, the numerical results shown in Fig. 11 indicate that the positive magnetic loading does aid the propagation of the crack, while the negative magnetic loading does impede it. Additionally, the
Fig. 9. $K'_I / K'^{\infty}_I$ versus the angle $\theta$.

Fig. 10. $K'_m / K'^{\infty}_m$ versus the angle $\theta$. 
corresponding larger magnitudes of the $G_{I}^{MA}/G_{I}^{M\infty}$ illustrated in Fig. 11 (c) and (d) than those in Fig. 11 (a) and (b) further suggest that the positive electric loading might aid the propagation of the crack.

6. Conclusions

From aforementioned numerical results and discussions, the following conclusions could be obtained

1. The proposed pseudo-traction–electric-displacement–magnetic-induction method (PTEDMIM) is really effective to solve the multiple crack interaction problems in magnetoelectroelastic materials;
2. The mechanical loading, the electric loading and the magnetic loading are actually coupled together to take effect on the crack interaction behaviors in magnetoelectroelastic material;
3. Similar as the positive and negative electric loadings play in piezoelectric materials, the positive and negative magnetic loadings bring an opposite influence on the crack interaction effect in magnetoelectroelastic materials. Moreover, the remote positive electric loading further enhances the crack interaction effect;
4. The crack interaction behavior in magnetoelectroelastic materials is not only dependent of the remote loading conditions, but also related to the geometric configuration of the cracks;
5. The neutral magnetic loading angles (NMLA) and the neutral electric angles (NELA) occur in the calculation results of the normalized mode I SIF $K_{I}^{MA}/K_{I}^{M\infty}$. Moreover, the numerical results show that the participation of the electric loading does not changes the NMLA and likewise, the participation of the magnetic loading does not changes the NELA either;
6. The numerical results of the $G_{ij}^{MA}/G_{ij}^{M∞}$ suggest that the positive magnetic loading generally aids the propagation of the crack, while the negative magnetic loading impedes it;
7. In collinear crack case, both the magnetic and electric loadings have not influence on the stress intensity factor, but they have a significant influence on the mechanical strain energy release rate.

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